FRACTIONS, FUNCTIONS AND FOLDING: COMMENTS ON THE THESIS OF JORIS NIEUWVELD

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This note is a summary of some novelties of the master thesis [4] of Joris Nieuwveld.

The theory of Mahler functions does not know many 'natural' examples that illustrate its achievements well; most of them originate from the automata theory and normally deal with functions that are lonely solutions of simple Mahler equations. One 'post-classical' example introduced by Dilcher and Stolarsky in [3] is the pair of $\{0,1\}$-power series $F(q)=1+q+q^2+q^5+\cdots$ and $G(q)=1+q+q^3+q^4+\cdots$, which solves the system of Mahler equations

$$F(q)=G(q^2)+qF(q^4), \quad G(q)=qF(q^2)+G(q^4). \tag{1}$$

In fact, if we assign the intitial values $F(0)=G(0)=1$ to a solution of (1) it is not hard to check that there is a unique power-series solution $F(q),G(q)$ given above. Notice that the functions $F(q),G(q)$ also satisfy individual Mahler equations (of order 2 with respect to $q\mapsto q^4$) but they are naturally chained with each other because

$$\operatorname{tr}\deg_{\mathbb{C}(q)}\mathbb{C}(q)[F(q),G(q),F(q^2),G(q^2)]=3 \tag{2}$$

as proven in [1, 2], with the algebraic relation

$$G(q)G(q^2)-qF(q)F(q^2)=1$$

given already in [3]. The series $F(q),G(q)$ are analytic in the unit disk $|q|<1$ and have the unit circle $|q|=1$ as a natural boundary of analyticity. It sounds perhaps sophisticated to ask whether there is still a reasonable way to discuss properties of the system (1) and its particular solutions on the unit circle and/or outside it (for example, in a spirit of [9]).

Reverting the role of 0 and ∞ in the case of (1) simply means investigating the pair of equations

$$\tilde F(q)=\tilde G(q^2)+q^{-1}\tilde F(q^4), \quad \tilde G(q)=q^{-1}\tilde F(q^2)+\tilde G(q^4).$$

This does not possess a nontrivial power-series solution but a (unique!) solution of the form $\tilde F(q)=q^{2/3}\hat F(q)$, $\tilde G(q)=q^{1/3}\hat G(q)$ with $\hat F(q)=1+q+\cdots$ and $\hat G(q)=1+q+\cdots$ analytic. In fact the substitution into the latter system reveals that

$$\hat F(q)=\hat G(q^2)+q\hat F(q^4), \quad \hat G(q)=\hat F(q^2)+q\hat G(q^4),$$

and a simple analysis shows that we have $\hat F(q)=\hat G(q)$, another $\{0,1\}$-power series, this common function under the name $H(q)$ satisfying $H(0)=1$ and

$$H(q)=H(q^2)+qH(q^4).$$

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Investigating the solution of this Mahler equation at ∞, $\tilde{H}(q) = \tilde{H}(q^2) + q^{-1}\tilde{H}(q^4)$, again reveals no solution in $\mathbb{C}[[q]]$ but in $q^{1/3}$; namely, the function $I(q) = q\tilde{H}(q^3)$ satisfying

$$I(q) = qI(q^2) + I(q^4)$$

and the initial condition $I(0) = 1$ is a unique $\{0,1\}$-power series. This $I(q)$ famously generates the Baum–Sweet sequence; in particular, $\operatorname{tr}\deg_{\mathbb{C}(q)} \mathbb{C}(q)[I(q), I(q^2)] = 2$ as demonstrated long time ago by Nishioka [5]. Nieuwveld extends the argument to prove that

$$\operatorname{tr}\deg_{\mathbb{C}(q)} \mathbb{C}(q)[H(q), H(q^2)] = 2,$$

so that — in a certain sense — there the transcendence degree (2) 'drops down' to 2 when we travel from solutions of the original system (1) to ones at ∞. Furthermore, Nieuwveld observes in [4] for the first time that

$$I(q) = qF(q^3) + G(q^3),$$

a nice connection between the Baum–Sweet sequence and Dilcher–Stolarsky functions.

The big portion of the work [4] is about continued fractions

$$a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots + a_{n-1} + \cfrac{b_n}{a_n}}}} = a_0 + \frac{b_1|}{|a_1} + \frac{b_2|}{|a_2} + \frac{b_3|}{|a_3} + \dots + \frac{b_n|}{|a_n}$$

and their limits

$$a_0 + \frac{b_1|}{|a_1} + \frac{b_2|}{|a_2} + \frac{b_3|}{|a_3} + \dots + \frac{b_n|}{|a_n} + \cdots$$

as $n \to \infty$ (when the latter exists!), related to the functions F, G, H and I. The above notation refers to 'irregular' continued fractions as opposed to the regular ones with all $b_i = 1$ and the shortcut notation

$$a_0 + \frac{1|}{|a_1} + \frac{1|}{|a_2} + \frac{1|}{|a_3} + \dots + \frac{1|}{|a_n} + \dots = [a_0, a_1, a_2, a_3, \dots, a_n, \dots]$$

for the 'regular' (classical) continued fractions. Explicitly, Nieuwveld considers the continued fractions

$$\lambda_n(q) = [q, q^2, q^4, q^8, \dots, q^{2^{n-1}}, q^{2^n}],$$
$$\lambda_n^+(q) = [q, q^2, q^4, q^8, \dots, q^{2^{n-1}}, q^{2^n} + 1]$$

and the 'upside-down' irregular ones

$$\rho_n(q) = 1 + \frac{q|}{|1} + \frac{q^2|}{|1} + \frac{q^4|}{|1} + \dots + \frac{q^{2^{n-2}}|}{|1} + \frac{q^{2^{n-1}}|}{|1}$$

and proves that

$$\lambda^+(q) = \lim_{n\to\infty} \lambda_n^+(q) = \frac{I(q)}{I(q^2)} \quad \text{and} \quad \rho(q) = \lim_{n\to\infty} \rho_n(q) = \frac{H(q)}{H(q^2)} \tag{3}$$

for $|q|<1$. This complements the continued fractions

$$\lim_{\substack{n\to\infty\\ n \text{ even}}} \lambda_n(q) = \frac{qF(q^3)}{G(q^6)} \quad \text{and} \quad \lim_{\substack{n\to\infty\\ n \text{ odd}}} \lambda_n(q) = \frac{G(q^3)}{q^2F(q^6)}$$

for $|q|<1$ of Dilcher and Stolarsky [3]. A culmination of the findings in [4] is the investigation of the limits in the domain $|q|>1$ and showing that, for $0<|q|<1$,

$$\lim_{\substack{n\to\infty\\ n \text{ even}}} \lambda_n(q) = q \lim_{\substack{n\to\infty\\ n \text{ even}}} \rho_n(q^{-3}) \quad \text{and} \quad \lim_{\substack{n\to\infty\\ n \text{ odd}}} \lambda_n(q) = q \lim_{\substack{n\to\infty\\ n \text{ odd}}} \rho_n(q^{-3}),$$

and for $|q|>1$,

$$\Bigl(\lim_{n\to\infty} \lambda_n^+(q) = \Bigr) \lim_{n\to\infty} \lambda_n(q) = q \lim_{n\to\infty} \rho_n(q^{-3})$$

whenever the right-hand side is defined (see (3)). This might be roughly stated as $\lambda(q)=q\rho(q^{-3})$ and interpreted as a connection between the objects a priori defined in the disjoint domains $|q|<1$ and $|q|>1$.

There are further remarkable features of the irregular continued fractions

$$\rho_{\mathrm{ev}}(q) = \lim_{\substack{n\to\infty\\ n \text{ even}}} \rho_n(q) \quad \text{and} \quad \rho_{\mathrm{od}}(q) = \lim_{\substack{n\to\infty\\ n \text{ odd}}} \rho_n(q)$$

for integral q with $|q|>1$. Namely, the corresponding *regular* continued fractions are folding-shaped similar to the famous continued fraction of $f(q)=q\sum_{n=0}^{\infty} q^{-2^n}$ (see [6, 8]). To state the result, we will associate with each finite continued fraction $[a_0,a_1,\dots,a_N]$ the word w, also recorded as $[a_0,a_1,\dots,a_N]$, and introduce the operations

$$-w = [-a_0,-a_1,\dots,-a_N], \quad \overleftarrow{w} = [a_N,a_{N-1},\dots,a_0]$$

and concatenation $[w,v]=[a_0,\dots,a_N,b_0,\dots,b_M]$ where $v=[b_0,b_1,\dots,b_M]$. In this notation, Theorem 2.2.4 [4] asserts that for the sequence of words $w_n=w_n(q)=[a_0,a_1,\dots,a_N]$ defined recursively by

$$w_n = [w_{n-2},(-1)^n q, -\overleftarrow{w_{n-2}}, (-1)^n q, w_{n-1}] \quad \text{for } n\ge 2$$

and $w_0=w_1=[\,]$ (the empty word), we have

$$\rho_n(q) = [s_n(q), w_n(q)], \qquad \text{where} \quad s_n(q) = \begin{cases} 1 & \text{for } n \text{ even},\\ q+1 & \text{for } n \text{ odd}. \end{cases}$$

In other words, when q is an integer, $|q|>1$, then all the partial quotients in the expansion of $\rho_n(q)$ (hence also of $\rho_{\mathrm{ev}}(q)$ and $\rho_{\mathrm{od}}(q)$), apart from the first one $s_n(q)$, are either q or $-q$. Notice that such regular continued fractions can be converted into 'traditional' ones with all partial quotients positive integers using the ripple lemma [7]; the resulted quotients still remain in a finite set of integers.

One way to see that the sequences w_n for n of same parity are non-periodic is to associate with a word $w=[a_0,a_1,\dots,a_N]$ (possibly infinite) the generating function $\mathrm{GF}(w)=\sum_n a_n q^n$ and consider the generating functions

$$\mathcal{F}(q) = \lim_{\substack{n\to\infty\\ n \text{ even}}} \mathrm{GF}(w_n(1)) \quad \text{and} \quad \mathcal{G}(q) = \lim_{\substack{n\to\infty\\ n \text{ odd}}} \mathrm{GF}(w_n(1)).$$

Then Nieuwveld shows that these $\{\pm 1\}$-power series satisfy the system of inhomogeneous Mahler equations

$$\mathcal{F}(q) = -q^2\mathcal{G}(q^2) + 1 + \frac{q}{1+q^2} - \frac{2q^4}{1+q^4}, \quad \mathcal{G}(q) = -\mathcal{F}(q^2) - \frac{q}{1+q^2} + \frac{2q^2}{1+q^4},$$

and are algebraically independent over $\mathbb{C}(q)$, so that

$$\operatorname{tr}\deg_{\mathbb{C}(q)} \mathbb{C}(q)[\mathcal{F}(q), \mathcal{G}(q), \mathcal{F}(q^2), \mathcal{G}(q^2)] = \operatorname{tr}\deg_{\mathbb{C}(q)} \mathbb{C}(q)[\mathcal{F}(q), \mathcal{G}(q)] = 2.$$

The thesis further discusses several variations of the themes and directions for future research; an interested reader is recommended to consult with the original source [4].

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