DIFFERENTIAL EQUATIONS, MIRROR MAPS AND ZETA VALUES

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Abstract. The aim of this work is an analytic investigation of differential equations producing mirror maps as well as giving new examples of mirror maps; one of these examples is related to (rational approximations to) $\zeta(4)$. We also indicate certain observations that might become a subject of further research.

The existence of this paper is due to the following observations. With Apéry's proof of the irrationality of $\zeta(2)$ and $\zeta(3)$ followed certain 2nd- and 3rd-order linear differential equations (see [Be1]–[Be3], [BP]). Doing a similar construction for $\zeta(4)$ resulted in a 5th-order differential equation. This equation was similar to the linear differential equations occuring in Calabi–Yau theory (except the order was 5 instead of 4). Computing the Lambert series of an analogue of the Yukawa coupling we got integer coefficients divisible by the square of the degree (not by the cube as in the Calabi–Yau case). Then we managed to pull back the 5th-order differential equation to one of order 4, which had all the properties of a Calabi–Yau equation. This part is the main objective of Sections 1–4. Quite recently we found two more examples relating fourth order differential equations to simultaneous rational approximations to di- and trilogarithms (see the end of Section 4).

Collecting known cases of the Calabi–Yau equations, 14 of those were classical hypergeometric and 15 remaining were discovered by V. V. Batyrev, D. van Straten et al in [BS1], [BS2], [Str], we tried to extend the list by application of the algorithm of creative telescoping due to Gosper and Zeilberger. This seems to be a pure machinery (cf. Section 5 below), in which we cannot predict any end. We also tried to figure out possible algebraic transformations between them and our new example related to $\zeta(4)$. We did not succeed in this, but doing that we found two other Calabi–Yau equations emanating from quadratic transformations of hypergeometric Calabi–Yau solutions of type ${}_4F_3$. This is the subject of Section 6. Section 7 describes another (rather rich!) possibilities for deriving fake 4th-order linear differential equations. Summarizing methods of Sections 6 and 7 resulted in new classes of transformations which we present in Section 8.

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The divisibility properties of the coefficients of the Lambert series for the Yukawa coupling $K(q)$ are equivalent to high-order Kummer congruences between the coefficients of the power series for $K(q)$. The latter coefficients can be realized as the number of fix points of iterates of a set map $T\colon X \to X$, where the number of orbits of order n is divisible by n^2. We figure out these very natural relations in Section 9.

Our final table (mostly represented in the joint contribution [AESZ, Table A]) contains more than 200 cases of Calabi–Yau equations (see also van Enckevort's electronic database [En], which provides more advanced knowledge, like instanton numbers up to 20, in some cases also the number of elliptic curves, monodromy matrices etc.). In almost all cases, the mirror map $z(q)$ divided by q seems to be a high power of a series with integer coefficients (cf. [LY]). For this we have no explanation at all, but we collect this experimental knowledge in another table [AESZ, Table B]. The last table [AESZ, Table C] presents a brief systematic guide to the main table from [AESZ, Table A], with entries ordered according to instanton numbers[1].

Finally, in Section 10, we find a 6th-order linear differential equation coming from the simultaneous approximations to $\zeta(3)$ and $\zeta(5)$, and also a similar 6th-order equation for the series $\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k}^6$. Both cases seem to admit arithmetic properties very close to those for Calabi–Yau cases, but for the latter series we have a new phenomenon—a free parameter. This fact does not look apparent: we briefly indicate some similar cases as well.

Acknowledgements. First of all we want to thank Duco van Straten, who has shown great interest in our work and suggested the use of Hadamard products. He and Christian van Enckevort contributed with finding several new Calabi–Yau equations, usually coming from geometry. Combining our examples and methods with those from [ES] resulted in the joint project [AESZ]. Frits Beukers gave us details of computations on [Be4] and inspiration to start this work; his comments helped us a lot to improve the first version of our manuscript. Helena Verrill communicated us a couple of new equations and Arne Meurman showed an algorithm to find the Hadamard product. Special gratitude is due to Noriko Yui who kindly proposed us to contribute the BIRS workshop volume on Calabi–Yau varieties and mirror symmetry.

Contents

[1] With the kind permit of our coauthors D. van Straten and C. van Enckevort, we include the last table from [AESZ] in the version of the present paper appearing in the BIRS workshop volume on Calabi–Yau varieties and mirror symmetry. The E-print version of this article at `http://arXiv.org/math.NT/0402386` also contains Section 10 and proofs of Propositions 1 and 3.

1. Maximal unipotent monodromy

Consider a linear differential equation

$$y^{(s)} + a_{s-1}(z)y^{(s-1)} + \cdots + a_1(z)y' + a_0(z)y = 0, \tag{1.1}$$

where the prime stands for the z-derivative and $a_0, a_1, \dots, a_{s-1}$ are meromorphic functions in variable z. Assuming that $z = 0$ is a regular singularity of the differential equation (1.1), we may write its coefficients in the form

$$a_{s-j}(z) = z^{-j}\tilde{a}_{s-j}(z), \qquad j = 1, \dots, s,$$

where the functions $\tilde{a}_{s-j}(z)$ are analytic at $z = 0$. The roots of the *indicial equation*

$$\lambda(\lambda-1)\cdots(\lambda-s+1) + \tilde{a}_{s-1}(0)\lambda(\lambda-1)\cdots(\lambda-s+2) + \cdots + \tilde{a}_1(0)\lambda + \tilde{a}_0(0) = 0$$

determine the *exponents* of the differential equation (1.1) at the point $z = 0$. Following [Mo], we will say that our differential equation (1.1) is of *maximal unipotent monodromy* (briefly, MUM) if its exponents at $z = 0$ are all zero. The Frobenius method (see, e.g., [In, Section 16.1]) gives one a constructive way for writing a basis of solutions to (1.1) in the form

$$\begin{gathered} y(z,\rho) = \sum_{n=0}^{\infty} A(n,\rho) z^{n+\rho} \ (\operatorname{mod} \rho^s) = y_0(z) + y_1(z)\rho + \cdots + y_{s-1}(z)\rho^{s-1}, \\ A(0,\rho) = 1 \ (\operatorname{mod} \rho^s) \end{gathered} \tag{1.2}$$

(in particular, $A(0,0) = y_0(0) = 1$), where by definition we set

$$z^\rho \ (\operatorname{mod} \rho^j) = 1 + \log z \cdot \rho + \frac{\log^2 z}{2} \cdot \rho^2 + \cdots + \frac{\log^{j-1} z}{(j-1)!} \cdot \rho^{j-1}.$$

The components $y_0, y_1, \dots, y_{s-1}$ in (1.2) define the *Frobenius basis* of the MUM differential equation (1.1) at $z = 0$.

2. 4th-order linear differential equations

Consider a linear homogeneous differential equation of order 4,

$$y^{(4)} + a_3(z)y^{(3)} + a_2(z)y'' + a_1(z)y' + a_0(z)y = 0, \tag{2.1}$$

where a_0, a_1, a_2, a_3 are rational functions in the variable z. Suppose that (2.1) is MUM and let y_0, y_1, y_2, y_3 be the Frobenius basis of solutions to (2.1). Set $t = y_1/y_0$.

Proposition 1 [Al2]. *In the above notation, the condition*

$$a_1 = \frac{1}{2}a_2a_3 - \frac{1}{8}a_3^3 + a_2' - \frac{3}{4}a_3a_3' - \frac{1}{2}a_3'' \tag{2.2}$$

is equivalent to any of the following two possibilities:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(y_2/y_0) = \frac{\exp\bigl(-\frac{1}{2}\int a_3(z)\,\mathrm{d}z\bigr)}{y_0^2\bigl(\frac{\mathrm{d}t}{\mathrm{d}z}\bigr)^3} \quad \textit{and} \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}(y_3/y_0) = t\frac{\mathrm{d}^2}{\mathrm{d}t^2}(y_2/y_0). \tag{2.3}$$

Proof. It can be easily checked that the functions

$$T_1 = \frac{y_1}{y_0}\ (=t), \quad T_2 = \frac{y_2}{y_0}, \quad T_3 = \frac{y_3}{y_0}$$

satisfy the 4th-order linear differential equation

$$T^{(4)} + b_3T^{(3)} + b_2T'' + b_1T' = 0, \tag{2.4}$$

where

$$b_1 = a_1 + 2a_2\frac{y_0'}{y_0} + 3a_3\frac{y_0''}{y_0} + 4\frac{y_0'''}{y_0},$$
$$b_2 = a_2 + 3a_3\frac{y_0'}{y_0} + 6\frac{y_0''}{y_0}, \qquad b_3 = a_3 + 4\frac{y_0'}{y_0}.$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t} = \left(\frac{\mathrm{d}t}{\mathrm{d}z}\right)^{-1}\frac{\mathrm{d}}{\mathrm{d}z} = \frac{1}{T_1'}\frac{\mathrm{d}}{\mathrm{d}z},$$

we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{y_2}{y_0}\right) = \frac{\mathrm{d}^2T_2}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{T_2'}{T_1'}\right) = \frac{1}{T_1'}\left(\frac{T_2''}{T_1'} - \frac{T_2'T_1''}{(T_1')^2}\right) = \frac{R}{(\mathrm{d}t/\mathrm{d}z)^3}, \tag{2.5}$$

where

$$R = T_1'T_2'' - T_1''T_2' = \begin{vmatrix} T_1' & T_2' \\ T_1'' & T_2'' \end{vmatrix}. \tag{2.6}$$

With the help of (2.4) we deduce

$$\begin{gathered} R' = \begin{vmatrix} T_1' & T_2' \\ T_1''' & T_2''' \end{vmatrix}, \quad R'' = \begin{vmatrix} T_1'' & T_2'' \\ T_1''' & T_2''' \end{vmatrix} - b_3R' - b_2R, \\ R''' = -b_3(R'' + b_3R' + b_2R) + b_1R - (b_3R')' - (b_2R)'. \end{gathered} \tag{2.7}$$

Therefore, the function R satisfies the 3rd-order differential equation

$$R''' + c_2R'' + c_1R' + c_0R = 0, \tag{2.8}$$

where

$$c_0 = b_2b_3 - b_1 + b_2', \quad c_1 = b_2 + b_3^2 + b_3', \quad c_2 = 2b_3.$$

We now claim that the function

$$\tilde{R} = \frac{1}{y_0^2}\exp\Bigl(-\frac{1}{2}\int a_3(x)\,\mathrm{d}x\Bigr)$$

also satisfies the linear differential equation (2.8) if and only if (2.2) holds. By differentiating we obtain

$$\begin{aligned}
\tilde{R}' &= -\biggl(\frac{a_3}{2}+2\frac{y_0'}{y_0}\biggr)\tilde{R},\\
\tilde{R}'' &= \biggl(\frac{a_3^2}{4}-\frac{a_3'}{2}+2a_3\frac{y_0'}{y_0}-2\frac{y_0''}{y_0}+6\frac{(y_0')^2}{y_0^2}\biggr)\tilde{R},\\
\tilde{R}''' &= \biggl(\frac{3a_3a_3'}{4}-\frac{a_3''}{2}-\frac{a_3^3}{8}+3\Bigl(a_3'-\frac{1}{2}a_3^2\Bigr)\frac{y_0'}{y_0}+3a_3\frac{y_0''}{y_0}-2\frac{y_0'''}{y_0}\\
&\qquad -9a_3\frac{(y_0')^2}{y_0^2}+18\frac{y_0'y_0''}{y_0^2}-24\frac{(y_0')^3}{y_0^3}\biggr)\tilde{R},
\end{aligned}$$

hence a necessary and sufficient condition for the function $\tilde{R}$ to satisfy (2.8) is the required one:

$$\biggl(\frac{a_2a_3}{2}-\frac{a_3^3}{8}+a_2'-\frac{3a_3a_3'}{4}-a_1\biggr)\tilde{R}=0.$$

It remains to compare the first three terms (i.e., coefficients of x^{-3}, x^{-2}, and x^{-1}) in order to verify the equality $\tilde{R}=R$.

We are now required to check the second equality in (2.3) provided (2.2) holds. Changing R to R_1 and T_2 to T_3 in (2.6), (2.7) we obtain that the function R_1 satisfies the differential equation (2.8). On the other hand, differentiating the product $T_1R=tR$ with the help of the differential equation (2.4) for T_1 and of (2.8) for R, we deduce that the function tR also satisfies (2.8). Comparing of the first three coefficients again yields $R_1=tR$, hence formulas (2.5) and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\biggl(\frac{y_3}{y_0}\biggr)=\frac{\mathrm{d}^2T_3}{\mathrm{d}t^2}=\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\frac{T_3'}{T_1'}\biggr)=\frac{R_1}{(\mathrm{d}t/\mathrm{d}z)^3}$$

complete the proof. □

If the differential equation (2.1) is MUM and assumption (2.2) holds, we define the *mirror map* $z=z(q)$ as the inverse of $q(z)=e^t=\exp(y_1/y_0)$ and the *Yukawa coupling* as

$$K(q)=N_0\cdot\frac{\mathrm{d}^2}{\mathrm{d}t^2}(y_2/y_0),$$

where N_0 is some integer different from 0. The interesting (*Calabi–Yau*) cases are those when $z(q)\in\mathbb{Z}[[q]]$ and

$$K(q)=\sum_{n=0}^{\infty}C_nq^n=N_0+\sum_{l=1}^{\infty}\frac{N_ll^3q^l}{1-q^l} \tag{2.9}$$

with

$$N_0=C_0\in\mathbb{Z} \qquad \text{and} \qquad N_l=\frac{1}{l^3}\sum_{d\mid l}\mu\Bigl(\frac{l}{d}\Bigr)C_d\in\mathbb{Z} \quad \text{for } l=1,2,\dots$$

and μ is the Möbius function (and, usually, $N_l>0$).

Proposition 2. *Under hypothesis* (2.2) *we have the identity for Wronskian determinants:*

$$\begin{vmatrix} y_0 & y_3 \\ y_0' & y_3' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Proof. The required identity is equivalent to $y_0^2(y_3/y_0)' = y_1^2(y_2/y_1)'$, i.e.

$$\left(\frac{y_3}{y_0}\right)' = \left(\frac{y_1}{y_0}\right)^2 \left(\frac{y_0}{y_1} \cdot \frac{y_2}{y_0}\right)' = t^2 \left(\frac{1}{t}\,\frac{y_2}{y_0}\right)'.$$

Since $\frac{\mathrm{d}}{\mathrm{d}z} = \frac{\mathrm{d}t}{\mathrm{d}z} \cdot \frac{\mathrm{d}}{\mathrm{d}t}$, verification of the last identity is equivalent to showing that $Y = 0$, where

$$Y = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y_3}{y_0}\right) - t^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{t}\,\frac{y_2}{y_0}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y_3}{y_0}\right) + \frac{y_2}{y_0} - t\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y_2}{y_0}\right).$$

But

$$\frac{\mathrm{d}Y}{\mathrm{d}t} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{y_3}{y_0}\right) - t\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{y_2}{y_0}\right) = 0$$

by Proposition 1. Therefore $Y = \mathrm{const}$, and verifying near $z = 0$ we get $Y = 0$ (all log-terms disappear). This completes the proof. □

3. Wronskian formalism

Consider an arbitrary pair $y(z), \tilde{y}(z)$ of linearly independent solutions of a 4th-order differential equation (2.1) (not necessarily MUM). Define the Wronskian determinant

$$Y = \begin{vmatrix} y & \tilde{y} \\ y' & \tilde{y}' \end{vmatrix}.$$

Proposition 3. *The function $Y = Y(z)$ satisfies a* 6*th-order linear differential equation with coefficients depending only on the coefficients of* (2.1) (*i.e., independent of the choice of the pair $y, \tilde{y}$*), *called the exterior square of* (2.1). *Moreover, this* 6*th-order differential equation reduces to a* 5*th-order differential equation* (*i.e., the coefficient of $Y^{(6)}$ is zero*) *if and only if condition* (2.2) *holds.*

Remark 1. In other words, condition (2.2) is equivalent to the exterior square having order 5. As pointed out by Beukers, by this equivalence properties (2.3) are the derived properties which are handy to calculate the Yukawa coupling. Nevertheless, we kept the original proof of Proposition 1 to make it independent of considerations in this section.

Remark 2. The clain of Proposition 3 may be proved in `Maple` by the `DEtools` command `exterior_power`.

Proof. We have

$$Y' = \begin{vmatrix} y & \tilde{y} \\ y'' & \tilde{y}'' \end{vmatrix}, \qquad u_3 = \begin{vmatrix} y & \tilde{y} \\ y''' & \tilde{y}''' \end{vmatrix},$$

$$u_4 = \begin{vmatrix} y' & \tilde{y}' \\ y'' & \tilde{y}'' \end{vmatrix}, \qquad u_5 = \begin{vmatrix} y' & \tilde{y}' \\ y''' & \tilde{y}''' \end{vmatrix}, \qquad u_6 = \begin{vmatrix} y'' & \tilde{y}'' \\ y''' & \tilde{y}''' \end{vmatrix}.$$

Using the differential equation (2.1) for $y, \tilde{y}$, we get the system

$$\begin{gathered} Y'' = u_3 + u_4, \qquad u_3' = u_5 - a_1 Y - a_2 Y' - a_3 u_3, \\ u_4' = u_5, \qquad u_5' = u_6 + a_0 Y - a_2 u_4 - a_3 u_5, \\ u_6' = a_0 Y' + a_1 u_4 - a_3 u_6. \end{gathered} \tag{3.1}$$

Then

$$\begin{aligned} Y''' &= u_3' + u_4' = u_5 - a_1 Y - a_2 Y' - a_3 u_3 + u_5 \\ &= 2u_5 - a_1 Y - a_2 Y' - a_3 (Y'' - u_4), \end{aligned} \tag{3.2}$$

and in the notation

$$U = Y''' + a_3 Y'' + a_2 Y' + a_1 Y$$

we may write (3.2) in the form

$$2u_5 = U - a_3 u_4. \tag{3.3}$$

Therefore

$$U' - a_3' u_4 - a_3 u_5 = (U - a_3 u_4)' = 2u_5' = 2u_6 + 2a_0 Y - 2a_2 u_4 - 2a_3 u_5.$$

Hence by (3.3)

$$U' + \frac{1}{2} a_3 U - 2a_0 Y = 2u_6 + \Big(\frac{1}{2} a_3^2 - 2a_2 + a_3'\Big) u_4$$

and after taking the derivative

$$U'' + \frac{1}{2} a_3 U' + \frac{1}{2} a_3' U - 2a_0 Y' - 2a_0' Y = 2u_6' + \Big(\frac{1}{2} a_3^2 - 2a_2 + a_3'\Big) u_5 + (a_3 a_3' - 2a_2' + a_3'') u_4.$$

Using (3.3) and (3.1) we write the last equality as $W = V u_4$, where

$$W = U'' + \frac{3}{2} a_3 U' + \Big(a_3 + \frac{1}{2} a_3^2 - \frac{1}{4} a_3^3\Big) U - 4a_0 Y' - 2(a_0' + a_0 a_3) Y$$

and

$$V = a_3'' - 2a_2' + \frac{3}{2} a_3 a_3' - a_2 a_3 + \frac{1}{4} a_3^2 + 2a_1.$$

The condition $V = 0$ is exactly (2.2); if it holds, we have $W = V u_4 = 0$, which is the required 5th-order differential equation. If $V \neq 0$, then differentiate $W = V u_4$ to get

$$W' = V' u_4 + V u_5 = V' u_4 + \frac{1}{2} V (U - a_3 u_4),$$

hence

$$V W' = \frac{1}{2} V^2 U + \Big(V' - \frac{1}{2} a_3 V\Big) W. \tag{3.4}$$

Equality (3.4) is the 6th-order linear differential equation for the function Y; the coefficient of $Y^{(6)}$ in this equation is $V \neq 0$. $\square$

From now on, we assume condition (2.2) for a given 4th-order MUM differential equation (2.1). To construct a 5th-order MUM differential equation we modify the above construction by taking

$$w(z) = zY = z \begin{vmatrix} y & \tilde{y} \\ y' & \tilde{y}' \end{vmatrix}.$$

Denote the resulting 5th-order differential equation for the function w:

$$w^{(5)} + b_4(z)w^{(4)} + b_3(z)w^{(3)} + b_2(z)w'' + b_1(z)w' + b_0(z)w = 0. \tag{3.5}$$

Remark. Clearly, condition (2.2) induces a certain condition for the coefficients of the differential equation (3.5). We leave the corresponding (rather boring) relation to the reader as a `Maple` exercise.

Proposition 4. *The differential equation* (3.5) *is MUM; its Frobenius basis is given by the following formulas:*

$$\begin{gathered} w_0 = z \begin{vmatrix} y_0 & y_1 \\ y_0' & y_1' \end{vmatrix}, \quad w_1 = z \begin{vmatrix} y_0 & y_2 \\ y_0' & y_2' \end{vmatrix}, \quad w_2 = z \begin{vmatrix} y_0 & y_3 \\ y_0' & y_3' \end{vmatrix} = z \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \\ w_3 = \frac{z}{2} \begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix}, \quad w_4 = \frac{z}{2} \begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix}. \end{gathered} \tag{3.6}$$

Remark. The inverse is not true in the following sense. If we take a MUM linear differential equation of order 5 with monodromy group O_5, then its 4th-order differential pullback is not necessarily MUM. An example to this is the 5th-order differential operator

$$D = \theta^5 - 6^6 z\big(\theta + \tfrac{1}{6}\big)\big(\theta + \tfrac{2}{6}\big)\big(\theta + \tfrac{3}{6}\big)\big(\theta + \tfrac{4}{6}\big)\big(\theta + \tfrac{5}{6}\big)$$

(which is a natural generalization of the 4th-order operator

$$D = \theta^4 - 5^5 z\big(\theta + \tfrac{1}{5}\big)\big(\theta + \tfrac{2}{5}\big)\big(\theta + \tfrac{3}{5}\big)\big(\theta + \tfrac{4}{5}\big)$$

corresponding to the Calabi–Yau differential equation).

Proof. The five functions in (3.6) are linearly independent solutions to (3.5). Developing their ($\log z$)-expansions (with the known structure of the Frobenius basis y_0, y_1, y_2, y_3 to (2.1)) show that they form the Frobenius basis of (3.5), hence the latter differential equation is MUM. $\square$

Corollary (Beukers's relations). *We have*

$$2w_0w_4 - 2w_1w_3 + w_2^2 = 0, \qquad 2w_0'w_4' - 2w_1'w_3' + {w_2'}^2 = 0. \tag{3.7}$$

Proof. Write

$$w_2^2 = z \begin{vmatrix} y_0 & y_3 \\ y_0' & y_3' \end{vmatrix} \cdot z \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

Then the first identity is trivial by expanding. In the same vein we obtain the identity for the functions $\tilde{w}_j$, in which the second line of the determinant contains y_i's replaced by y_i''s. Since $w_j' = w_j/z + \tilde{w}_j$, we finally arrive at the second identity in (3.7). □

Remark. Beukers proved the above relations for a concrete example of the 5th-order differential equation, which we discuss in the next section.

In the case of the 5th-order MUM differential equation (3.5) we can define analogues of a mirror map and a Yukawa coupling. Namely, we can define $\tilde{z}(q)$ as the inverse of $q(z) = \exp(w_1/w_0)$ and

$$\tilde{K}(q) = \tilde{N}_0 \cdot \frac{\mathrm{d}^2}{\mathrm{d}t^2}(w_2/w_0), \qquad t = w_1/w_0.$$

Developing the last q-series as the Lambert series,

$$\tilde{K}(q) = \tilde{N}_0 + \sum_{l=1}^{\infty} \frac{l^2 \tilde{N}_l q^l}{1-q^l} \tag{3.8}$$

(we put l^2 instead of l^3 here), we have the following experimental observation: $\tilde{z}(q) \in K[[q]]$ and $\tilde{N}_l \in K$, where K is either $\mathbb{Z}$ or $\mathbb{Z}[1/p]$ for a certain prime p (the possibilities $p = 3, 5, 7, 23$ occur).

4. Strange integrality related to $\zeta(4)$

In the work [Zu1], the following 3-term polynomial recursion is given:

$$\begin{aligned}(n+1)^5 A_{n+1} &- 3(2n+1)(3n^2+3n+1)(15n^2+15n+4)A_n \\ &- 3n^3(3n-1)(3n+1)A_{n-1} = 0 \qquad \text{for} \quad n \geqslant 1.\end{aligned}$$

If we take two linearly independent solutions $\{A_n\}$ and $\{B_n\}$ to the above recursion given by the initial data

$$A_0 = 1, \quad A_1 = 12, \qquad \text{and} \qquad B_0 = 0, \quad B_1 = 13,$$

then

$$\lim_{n\to\infty} \frac{B_n}{A_n} = \zeta(4) = \frac{\pi^4}{90}$$

(see [Zu1] for details). This recursion was previously proved in [Co] and [So] but without any indication of arithmetic properties of the sequences $\{A_n\}$ and $\{B_n\}$. In [Zu1], it is proved that

$$D_n A_n, D_n^5 B_n \in \mathbb{Z}, \qquad \text{where} \quad D_n = \text{the least common multiple of } 1, 2, \dots, n,$$

and conjectured some stronger inclusions that were finally proved in [KR]. In particular, from [KR] we have the inclusions

$$A_n, D_n^4 B_n \in \mathbb{Z},$$

and even the following explicit formula (see also [Zu4]):

$$A_n = \sum_{j,k} \binom{n}{j}^2 \binom{n}{k}^2 \binom{n+j}{n} \binom{n+k}{n} \binom{j+k}{n},$$

where the binomial coefficients $\binom{a}{b}$ are zero if $b < 0$ or $a < b$.

A natural object related to the above recursion is the 5th-order MUM differential equation

$$\begin{aligned} &z^5(27z^2+270z-1)w^{(5)} + z^4(405z^2+3375z-10)w^{(4)} \\ &\quad + z^3(1752z^2+11502z-25)w^{(3)} + z^2(2412z^2+11259z-15)w'' \\ &\quad + z(816z^2+2130z-1)w' + 12z(2z+1)w = 0. \end{aligned} \tag{4.1}$$

Denote by w_0, w_1, w_2, w_3, w_4 the Frobenius basis of (4.1) near $z = 0$, so that

$$w_0 = w_0(z) = \sum_{n=0}^{\infty} A_n z^n = 1 + 12z + 804z^2 + 88680z^3 + 12386340z^4 + \cdots .$$

It was pointed out by F. Beukers [Be4] that the (Zariski closure of the) monodromy group of the differential equation (4.1) turns out to be O_5. This result (as well as identities (3.7) for the latter functions w_js) are consequences of the following statement.

Proposition 5. *The* 5*th-order MUM differential equation* (4.1) *is constructed from the* 4*th-order MUM differential equation* (2.1) *with*

$$\begin{aligned} a_3 &= \frac{6(36z^2+315z-1)}{z(27z^2+270z-1)}, \\ a_2 &= \frac{10530z^4+180387z^3+759417z^2-4671z+7}{z^2(27z^2+270z-1)^2}, \\ a_1 &= \frac{96228z^6+2421009z^5+18416565z^4+49339854z^3-450054z^2+1107z-1}{z^3(27z^2+270z-1)^3}, \\ a_0 &= \frac{\begin{array}{l} 3(8748z^7+253692z^6+10875303z^5-37601010z^4 \\ \qquad +13643328z^3+432135z^2+13224z+11) \end{array}}{z^3(27z^2+270z-1)^4} \end{aligned}$$

by the algorithm of Section 3.

Proof. By direct computation (using `Maple`). □

Remark. The original proof by Beukers [Be5] of relations (3.7) for the differential equation (4.1) follows the principle 'the proof is trivial once has found the explicit formula'. We reproduce the arguments from [Be5] here. The symmetric square of the 5th order equation (4.1) has order 14, one order less than expected 15. This means that there should be a homogeneous quadratic relation between the solutions $w_0, \dots, w_4$ of (4.1). It suffices to find a relation which vanishes at $z = 0$ of order one higher than the local exponents given by the symmetric square of order 14. (Clearly, any non-trivial relation satisfies the 14th order equation; if we find one with one vanishing order higher than predicted by this 14th order equation, it must be the trivial solution.) If we replace the w_js by their derivatives, we get quadratic forms that form a one-dimensional monodromy representation. Looking at the local exponents we can infer that they should be rational functions with poles only at infinity. Hence the expressions are polynomial of degree again given by the local exponents at infinity of the symmetric square.

Here are some observations related to the 5th-order differential equation (4.1) and its 4th-order pullback given in Proposition 5 (we denote by y_0, y_1, y_2, y_3 the Frobenius basis of the latter differential equation).

Observation 1. We have $y_0 \in \mathbb{Z}[1/2][[z]]$ but not $y_0 \in \mathbb{Z}[[z]]$, while $z(q), K(q) \in \mathbb{Z}[[q]]$ and for $K(q)$ we have the Lambert expansion (2.9) with $N_l \in \mathbb{Z}$ but $N_l/N_0 < 0$.

Observation 2. We have $w_0 = u_0^2$, where

$$u_0 = 1 + 6z + 384z^2 + 42036z^3 + 5867226z^4 + \cdots \in \mathbb{Z}[[z]].$$

Observation 3. One has the following generalized Beukers's relations: for $k \geqslant 2$

$$2w_0^{(k)}w_4^{(k)} - 2w_1^{(k)}w_3^{(k)} + (w_2^{(k)})^2 = \sum_{\nu} c_{k,\nu}(9z)^{\nu},$$

where $c_{k,\nu} \in \mathbb{Z}$.

Observation 4. Put $t = w_1/w_0$, $q(z) = e^t = z + O(z^2)$ and denote by $\tilde{z}(q)$ its inverse. Then $q(z) \in \mathbb{Z}[[z]]$ and, hence, $\tilde{z}(q) \in \mathbb{Z}[[q]]$. Moreover,

$$\begin{aligned} q(z) &= z(1 + 15z + 1145z^2 + \cdots)^3, \\ \tilde{z}(q) &= q(1 - 15q - 245q^2 - 20138q^3 - 2043703q^4 + \cdots)^3. \end{aligned}$$

Observation 5. Expand w_0 in Lambert series:

$$2w_0(z) = 2 + \sum_{n=1}^{\infty} \frac{n^3 L_n z^n}{1 - z^n}.$$

Then $L_n \in \mathbb{Z}$ and $L_n > 0$. What is a enumerative meaning of the numbers L_n (i.e., do they count some geometric objects)?

Observation 6. Write

$$\tilde{K}(q) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(w_2/w_0) = 1 + \sum_{l=1}^{\infty} \frac{l^2 \tilde{N}_l q^l}{1 - q^l}.$$

Then $\tilde{N}_l \in \mathbb{Z}$ and $\tilde{N}_l > 0$. What is a enumerative meaning of the numbers $\tilde{N}_l$ (i.e., do they count some geometric objects)?

Observation 7 (Kummer supercongruences). If we write w_0 and $\tilde{K}$ as functions of z,

$$w_0(z) = \sum_{n=0}^{\infty} A(n) z^n, \qquad \tilde{K}(q(z)) = \sum_{n=0}^{\infty} C(n) z^n,$$

then

$$A(np^r) \equiv A(np^{r-1}) \pmod{p^{3r}}, \qquad C(np^r) \equiv C(np^{r-1}) \pmod{p^{2r}},$$

for primes p and $(n,p)=1$; $r = 1, 2, \dots$.

Recently the second author [Zu5] (see also Theorem 4 in [Zu2]), when approximating $\mathrm{Li}_3(z)$ (see definition (9.3) below), found two more *fourth order* examples related to simultaneous rational approximations to $\zeta(2)$ and $\zeta(3)$. About the same time the first author discovered compact formulas

$$A_n = \sum_{k,l} \binom{n}{k}^2 \binom{n}{l}^2 \binom{k+l}{l} \binom{n+k}{n} \qquad n = 0, 1, 2, \dots, \tag{4.2}$$

and

$$A_n = \binom{2n}{n} \sum_k \binom{n}{k}^2 \binom{n+k}{n} \binom{n+2k}{n}, \qquad n = 0, 1, 2, \dots,$$

for generating series $y_0(z) = \sum_{n=0}^{\infty} A_n z^n$ satisfying exactly the same differential equations. These examples are cases #195 and #209 in [AESZ, Table A]. The coefficients (4.2) is of the form similar to those in [BS2] (cf. case #26), and van Straten has soon confirmed that equation #195 indeed 'comes from geometry'. Thus we have a direct link between number theory and geometry.

5. Strategy of finding new Calabi–Yau equations

How much do we know about Calabi–Yau differential equations? We discovered a list of 29 interesting cases taken from different papers on the subject; these cases initiated our systematization of Calabi–Yau equations and started the table in [AESZ][2]. The first 14 cases in the table correspond to all known hypergeometric Calabi–Yau equations, while the origin of the remaining 15 known cases is due to certain variations of multiple hypergeometric series (or multiple binomial sums). The latter circumstance prompted us to look on those linear MUM differential equations, whose analytic solutions $y_0(z) \in \mathbb{Z}[[z]]$ are single or multiple sums of certain hypergeometric terms. There is an algorithmic way, which is due to Gosper and Zeilberger known as *creative telescoping*, to find polynomial recursions for such sums [PWZ]. Unfortunately, the algorithm for multiple summations (being a part of the general *WZ-theory* developed by H. Wilf and D. Zeilberger [WZ]) is not implemented in `Maple` but solving auxiliary systems of linear equations is. The recursions may be easily transfered to linear differential equations satisfied by $y_0(z)$. That was our general recipe for verifying known cases #15–29, and then we continued our experiments by constructing several

[2]Unfortunately, we are not quite sure that the nice arithmetic properties of mirror maps and Yukawa couplings are really proved in all presented cases! In this sense, [AESZ, Table A] involves cases that are only expected to be 'arithmetically nice'.

new cases, also in combination with other hypergeometric methods—quadratic transformations (Section 6), Hadamard products (Section 7) and other transformations (Section 8). In several cases the algorithm of creative telescoping led to reducible differential equations, but then we used `DEtools` implemented in `Maple` to factorize corresponding differential operators.

The way *a là Apéry* of producing new examples of the mirror maps and Yukawa couplings, decribed in the previous section seems to be very occasional: we do not really have other examples of rational approximations to 'interesting' constants (but see Section 10). Nevertheless, the method of finding Calabi–Yau equations by pulling back some 'nice' 5th-order linear differential equations works in several other situations as well. We stress that the 5th- to 4th-order reduction did not appear before in connection with problems of Calabi–Yau manifolds.

Our table in [AESZ] extends the list of 29 known cases and contains a sporadic list of more than two hundred Calabi–Yau (conjecturally) differential equations. The table is long but we have found many more equations that we decided not to include (not to stretch the reader's patience too far!). In particular, we have many examples of type $A_n = \sum_k (-1)^k 2^{n-2k} \binom{n}{2k}\binom{2k}{k}(\,\cdot\,)$. The most unfortunate thing seems to be the fact that we cannot see any reasonable end of our list and, therefore, we do not expect to be much criticized by the reader for missing a hundred or more other interesting cases.

The analysis of [AESZ, Table A] shows that, besides the 14 hypergeometric functions (cases #1–14), we have a plenty of other 'simple' Calabi–Yau differential equations of type

$$\theta^4 - c_1 z(a\theta^2 + a\theta + b)P(\theta) + c_2 z^2 Q(\theta),$$

where $P(\theta)$ is of the form $(2\theta+1)^2$, $(3\theta+1)(3\theta+2)$, $(4\theta+1)(4\theta+3)$ or $(6\theta+1)(6\theta+5)$, and $Q(\theta)$ splits into linear factors of similar shapes. Here $a \in \{2, 3, 5, 7, 8, 9, 11, 13, 17\}$ and c_1, c_2 are usually powers of 2 and (or) 3. We also have 5th-order differential equations similar to the the 4th-order in 2. Their 4th-order pullbacks (derived by the method of Section 3) are much more complicated. Case #32 seems to be even more complicated:

$$\begin{aligned}\theta^5 - 3z(2\theta+1)(3\theta^2+3\theta+1)(15\theta^2+15\theta+4)\\ - 3z^2(\theta+1)^3(3\theta+2)(3\theta+4)\end{aligned}$$

(this is the differential equation that started this paper). Then there are the cases, when both the 4th-order differential equation and the 5th-order equation (obtained by the wronskian construction) are very complicated. The corresponding linear differential operators usually appear as factors of higher order differential operators, for which we have no pattern at all.

One can suspect that the equations in the list from [AESZ, Table A] are Picard–Fuchs equations of one-dimensional families of Calabi–Yau threefolds. In many cases, such as the hypergeometric ones and the Hadamard products, this should not be hard to see. For other cases, such as the one related to $\zeta(4)$, this fact will be quite nontrivial, but perhaps all the more interesting. This is a reason of why the equations are named Calabi–Yau equations.

Remark. We are quite surprised by the enomous amount of the 4th-order linear differential equations with (expectively) nice arithmetic properties. After discovering equation #9, the first author for a long time thought that there were no more examples but #1–14. Then after finding the papers [BS1], [BS2] and examples #15–28 there, it was still hard to find any new cases. The use of the Gosper–Zeilberger algorithm changed this drastically. For a while we thought that whenever A_n was a binomial sum (even multiple) and it gave a 4th-order MUM differential equation, then we automatically got an integer mirror map $z(q)$ and the Yukawa coupling $K(q)$ as in (2.9) with integers N_l. There is, however, the following example: the generating series $y_0 = \sum_{n=0}^{\infty} A_n z^n$, where

$$A_n = \sum_k (-1)^k 4^{n-4k} \binom{n}{4k} \frac{(5k)!}{k!^5},$$

satisfies a 4th-order MUM differential equation with polynomial coefficients of degree 8; the equation produces the non-integer mirror map $z(q)$ (actually, $z(q) \in \mathbb{Z}[1/2][[q]]$) but the Yukawa coupling has integers N_l. There are more examples, also for the cases of non-integers N_l; we indicate one in Section 7 below.

6. Quadratic transformations and mirror maps

A purely analytic, hypergeometric machinery—quadratic and higher-order transformations—may be put forward for constructing new examples from the old ones. The method powerfully works in the case of 2nd- and 3rd-order differential equations, when we have to substitute a suitable modular function in place of variable z; see, for example, [HM].

Our starting point is the existence of quadratic transformations for ${}_2F_1$- and ${}_3F_2$-series, like

$${}_2F_1\left(\begin{matrix} a, \quad b \\ 1+a-b \end{matrix} \,\middle|\, z\right) = (1-z)^{-a} \cdot {}_2F_1\left(\begin{matrix} \frac12 a, \frac12 + \frac12 a - b \\ 1+a-b \end{matrix} \,\middle|\, -\frac{4z}{(1-z)^2}\right) \tag{6.1}$$

due to Gauß and

$$\begin{aligned} &{}_3F_2\left(\begin{matrix} a, \quad b, \quad c \\ 1+a-b, 1+a-c \end{matrix} \,\middle|\, z\right) \\ &\quad = (1-z)^{-a} \cdot {}_3F_2\left(\begin{matrix} \frac12 a, \ \frac12 + \frac12 a, \ 1+a-b-c \\ 1+a-b, \ 1+a-c \end{matrix} \,\middle|\, -\frac{4z}{(1-z)^2}\right) \end{aligned} \tag{6.2}$$

due to Whipple. Substituting $a = b = c = \frac12$ in them, we get algebraic relations between different mirrors that are necessarily *modular* maps in these cases. These quadratic transformations have a natural generalization to the case of higher dimensional hypergeometric series (see [Zu3], Sections 6 and 7). Similar but particular results are given in the following statement.

Proposition 6. *The following quadratic transformations are available for ${}_4F_3$-series:*

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix} a,\ b,\ c,\ d \\ 1+a-b,\ 1+a-c,\ 1+a-d \end{matrix}\,\middle|\, z\right)\\
&\quad= \frac{1}{(1+z)^a}\sum_{n=0}^{\infty}\frac{(\frac12 a)_n\,(\frac12+\frac12 a)_n}{(1+a-b)_n\,(1+a-c)_n}\left(\frac{4z}{(1+z)^2}\right)^n\\
&\qquad\times\sum_{\nu=0}^{n}\frac{(b)_\nu\,(c)_\nu\,(1+a-b-c)_{n-\nu}}{\nu!\,(n-\nu)!\,(1+a-d)_\nu} \qquad (6.3)\\
&\quad= \frac{1}{(1-z)^a}\sum_{n=0}^{\infty}\frac{(\frac12 a)_n}{(1+a-b)_n}\left(-\frac{4z}{(1-z)^2}\right)^n\sum_{\mu=0}^{n}\frac{(b)_\mu\,(\frac12+\frac12 a)_\mu\,(\frac12+\frac12 a-b)_{n-\mu}}{(n-\mu)!\,(1+a-c)_\mu}\\
&\qquad\times\sum_{\nu=0}^{\mu}\frac{(c)_\nu}{\nu!\,(\mu-\nu)!\,(1+a-d)_\nu}(-1)^\nu. \qquad (6.4)
\end{aligned}
$$

Proof. Writing

$$
\begin{aligned}
&{}_4F_3\left(\begin{matrix} a,\ b,\ c,\ d \\ 1+a-b,\ 1+a-c,\ 1+a-d \end{matrix}\,\middle|\, z\right)\\
&\quad=\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n\,(c)_n}{n!\,(1+a-b)_n\,(1+a-c)_n}(-z)^n\cdot{}_2F_1\left(\begin{matrix} -n,\ a+n \\ 1+a-d \end{matrix}\,\middle|\,1\right)\\
&\quad=\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{\nu!\,(1+a-d)_\nu}\sum_{n=\nu}^{\infty}\frac{(a)_{\nu+n}\,(b)_n\,(c)_n}{(n-\nu)!\,(1+a-b)_n\,(1+a-c)_n}(-z)^n\\
&\quad=\sum_{\nu=0}^{\infty}\frac{(a)_{2\nu}(b)_\nu(c)_\nu}{\nu!\,(1+a-b)_\nu\,(1+a-c)_\nu\,(1+a-d)_\nu}z^\nu\\
&\qquad\times{}_3F_2\left(\begin{matrix} a+2\nu,\ b+\nu,\ c+\nu \\ 1+a-b+\nu,\ 1+a-c+\nu \end{matrix}\,\middle|\,-z\right). \qquad (6.5)
\end{aligned}
$$

To the latter ${}_3F_2$-series we apply the quadratic transformation (6.2) and reorder summations to get (6.3).

For the proof of (6.4), we proceed as in (6.5) to deduce

$$
\begin{aligned}
&{}_3F_2\left(\begin{matrix} a,\ b,\ c \\ 1+a-b,\ 1+a-c \end{matrix}\,\middle|\, z\right)\\
&\quad=\sum_{\lambda=0}^{\infty}\frac{(a)_{2\lambda}(b)_\lambda}{\lambda!\,(1+a-b)_\lambda\,(1+a-c)_\lambda}z^\lambda\cdot{}_2F_1\left(\begin{matrix} a+2\lambda,\ b+\lambda \\ 1+a-b+\lambda \end{matrix}\,\middle|\,-z\right). \qquad (6.6)
\end{aligned}
$$

To the inner ${}_2F_1$-series we now apply the transformation (6.1) to get, as before, the double series for the left-hand side of (6.6). Then substituting the resulting formula for the ${}_3F_2$-series into (6.5) we arrive at the desired identity (6.4). ☐

Plugging in $a=b=c=d=\frac12$ we obtain

$$\sum_{n=0}^\infty \binom{2n}{n}^4\left(\frac{z}{2^8}\right)^n = (1+z)^{-1/2}\sum_{n=0}^\infty A_n^{(+)}\left(\frac{z}{2^8(1+z)^2}\right)^n$$
$$= (1-z)^{-1/2}\sum_{n=0}^\infty A_n^{(-)}\left(-\frac{z}{2^8(1-z)^2}\right)^n,$$

where

$$A_n^{(+)} = \frac{(4n)!}{(2n)!\,n!^2}\sum_{\nu=0}^n 2^{2(n-\nu)}\binom{2\nu}{\nu}^2\binom{2(n-\nu)}{n-\nu}, \tag{6.7}$$

$$A_n^{(-)} = 2^{2n}\frac{\prod_{j=0}^{n-1}(1+4j)}{n!}\sum_{\mu=0}^n 2^{4(n-\mu)}\binom{2\mu}{\mu}\frac{\prod_{j=0}^{n-\mu-1}(1+4j)}{(n-\mu)!}\,\frac{\prod_{j=0}^{\mu-1}(3+4j)}{\mu!}$$
$$\times\sum_{\nu=0}^{\mu}2^{2(\mu-\nu)}\binom{2\nu}{\nu}\binom{\mu}{\nu}(-1)^\nu. \tag{6.8}$$

The resulting series $\sum_{n=0}^\infty A_n^{(+)}z^n$ and $\sum_{n=0}^\infty A_n^{(-)}z^n$ are solutions of 4th-order linear differential equations. They gave two new examples (cases #30 and #31) of differential equations producing mirror maps and Yukawa couplings with desired integrality properties. It seems to be an interesting problem to find quadratic and high-order transformations for other cases from [AESZ, Table A].

Besides hypergeometric transformations, there is also an ‘obvious’ way to produce infinitely many linear MUM differential equations starting with the only one such example (1.1).

Proposition 7. *Let $y=y(z)$ be a generic solution of equation (1.1) of order s with coefficients in $\mathbb{C}(z)$ and let $u=u(z)$ be any function satisfying $u'/u\in\mathbb{C}(z)$. Then the function $w=y/u$ is a solution of the linear differential equation of order s with rational coefficients, and this equation depends only on (1.1) but not on its solution y.*

Proof. Substituting $y=uw$ in (1.1) we get the equation

$$(uw)^{(s)}+a_{s-1}(z)(uw)^{(s-1)}+\cdots+a_1(z)(uw)'+a_0(z)uw=0 \tag{6.9}$$

in w unknown. Developing the derivatives by the formulas

$$(uw)^{(k)}=\sum_{j=0}^k\binom{k}{j}u^{(j)}w^{(k-j)},\qquad k=1,2,\dots,s,$$

and dividing the left-hand side in (6.9) by u, we obtain the sth-order linear differential equation for w with rational coefficients since $u^{(j)}/u\in\mathbb{C}(z)$ for any $j=0,1,2,\dots$. □

By Proposition 7, taking a known example of a Calabi–Yau equation with the corresponding series $y_0=\sum_{n=0}^\infty A_nz^n\in\mathbb{Z}[[z]]$ and choosing any $u\in 1+z\mathbb{Z}[[z]]$ with the property $u'/u \in \mathbb{Q}(z)$, we see that $w_0 = y_0/u$ is in $\mathbb{Z}[[z]]$ and $w = y/u$ also satisfies a Calabi–Yau equation, with the same mirror map $z(q)$ and Yukawa coupling $K(q)$, since $w_j/w_0 = y_j/y_0$ for $j = 1, 2, 3$. For instance, the choice $u = \sqrt{1-4pz}$, where p is an arbitrary integer, leads to an infinite family of Calabi–Yau equations for the sequences

$$\widehat{A}_n = \sum_{k=0}^{n} p^{n-k} \binom{2n-2k}{n-k} A_k.$$

Clearly, examples of such type are not presented in Appendix A.

We also found experimentally another (doubly) infinite classes of Calabi–Yau differential equations depending on positive integer parameters p and r.

Proposition 8. *Given A_n, one of the cases presented in* [AESZ, Table A] *with the corresponding Yukawa coupling $K(q)$, define*

$$\widehat{A}_n = \sum_{k} p^{n-rk} \binom{n}{rk} A_k.$$

Then $\widehat{y}_0 = \sum_{n=0}^{\infty} \widehat{A}_n z^n$ satisfies a 4th-order MUM differential equation with Yukawa coupling $\widehat{K}(q) = K(q^r)$.

Remark. It can occur that the mirror map $\widehat{z}(q)$ does not lie in $\mathbb{Z}[[q]]$ (but the denominators of its coefficients are necessarily divisors of powers of r).

Proof. We only give a sketch of the proof, without indicating computational details. Suppose that D is the linear 4th-order differential operator annihilating the given series $y_0(z) = \sum_{n=0}^{\infty} A_n z^n$ and y_0, y_1, y_2, y_3 is the Frobenius basis of the differential equation $Dy = 0$. Change the variable $z \mapsto Z(z) = \big(z/(1-pz)\big)^r$; then substituting

$$\delta = z\frac{\mathrm{d}}{\mathrm{d}z} = \frac{r}{1-pz} \cdot Z\frac{\mathrm{d}}{\mathrm{d}Z}$$

into the differential equation $Dy = 0$ leads to a new 4th-order MUM differential equation that annihilates $Y_0(z) = y_0(Z(z))$. There is no difficulty in computing the Frobenius basis of the new differential equation (cf. (1.2)): it is

$$Y_j(z) = r^{-j} y_j(Z(z)), \qquad j = 0, 1, 2, 3.$$

Therefore, $T(z) = Y_1(z)/Y_0(z) = \frac{1}{r}t(Z(z))$ and the new mirror map $Z(q)$ is related to the old one $z(q)$ in accordance with the formula

$$z(q^r) = \left(\frac{Z(q)}{1-pZ(q)}\right)^r.$$

Finally,

$$\frac{\mathrm{d}^2}{\mathrm{d}T^2}\left(\frac{Y_2}{Y_0}\right) = K(q^r).$$

Applying now the composition

$$y(z) \mapsto \widetilde{y}(z) = \frac{1}{1-pz} \cdot y(Z(z)) = \frac{1}{1-pz} \cdot y\biggl(\Bigl(\frac{z}{1-pz}\Bigr)^r\biggr)$$

with the help of Proposition 7 we conclude that the differential equation for $\widetilde{y}(z)$ is of order 4 and MUM, and the corresponding Yukawa coupling is $K(q^r)$. It remains to verify that

$$\widetilde{y}_0(z) = \sum_{n=0}^{\infty} \widetilde{A}_n z^n = \frac{1}{1-pz} \sum_{n=0}^{\infty} A_n \Bigl(\frac{z}{1-pz}\Bigr)^{rn} = \frac{1}{1-pz} \cdot y_0\biggl(\Bigl(\frac{z}{1-pz}\Bigr)^r\biggr). \qquad \square$$

Remark. As pointed out to us by Beukers, Propositions 7 and 8 might be discarded by choosing a normalization for Calabi–Yau differential equations. Natural requirements are as follows:

(a) the smallest local exponent at the finite singularities is zero;
(b) the point $z = \infty$ is a singularity.

However, the propositions remain useful if these conditions are abandoned.

7. Hadamard products

Using the algorithm of creative telescoping we found almost a hundred new Calabi–Yau differential equations. Then D. van Straten suggested that using Hadamard products of solutions to 'nice' 2nd-order equations could give required 4th-order equations. Surprisingly, this was also the case for many of the examples we had found just by accident.

Let

$$u = \sum_{n=0}^{\infty} b_n z^n, \qquad v = \sum_{n=0}^{\infty} c_n z^n$$

be two D-finite (i.e., satisfying a linear differential equation of finite order with polynomial coefficients) power series. Then the Hadamard product

$$y = \sum_{n=0}^{\infty} a_n z^n = u * v = \sum_{n=0}^{\infty} b_n c_n z^n \tag{7.1}$$

is also D-finite (see [Sta, p. 194]). If D_u and D_v are linear differential operators annihilating u and v, respectively, by $D_u * D_v$ we denote the differential operator annihilating (7.1). We do not know a general algorithm for computing $D_u * D_v$, but for a given pair of operators D_u, D_v the problem is easily solved by linear algebra arguments. We have found about 30 second order MUM differential equations coming from binomial coefficients; Zagier's manuscript [Za] provides us with 36 second order examples with 'nice' arithmetic properties, although a closed binomial-sum formula is not known in all cases. Consider the following examples for which we were successful in finding the differential equations for their Hadamard products:

(a) $A_n = \sum_{k=0}^{n} \binom{n}{k}^3, \qquad D = \theta^2 - z(7\theta^2 + 7\theta + 2) - 8z^2(\theta+1)^2;$

(b) $A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}, \qquad D = \theta^2 - z(11\theta^2 + 11\theta + 3) - z^2(\theta+1)^2;$

(c) $A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}, \qquad D = \theta^2 - z(10\theta^2 + 10\theta + 3) + 9z^2(\theta+1)^2;$

(d) $A_n = \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}\binom{2n-2k}{n-k}, \qquad D = \theta^2 - 4z(3\theta^2 + 3\theta + 1) + 32z^2(\theta+1)^2;$

(e) $A_n = \sum_{k=0}^{n} 4^{n-k}\binom{2k}{k}^2\binom{2n-2k}{n-k}, \qquad D = \theta^2 - 4z(8\theta^2 + 8\theta + 3) + 256(\theta+1)^2.$

Taking all possible Hadamard products of these examples (including squares) we get 15 new differential equations. Squares will be of degree 5 and the others of degree 8. The products are listed in #100–107, 113–115 and 120–123.

We may also extend the above list by the following examples (f)–(h) due to D. Zagier [Za] and (i), (j) due to C. van Enckevort and D. van Straten [ES]:

(f) $A_n = \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \frac{(3k)!}{k!^3}, \qquad D = \theta^2 - 3z(3\theta^2 + 3\theta + 1) + 27z^2(\theta+1)^2;$

(g) $\qquad D = \theta^2 - z(17\theta^2 + 17\theta + 6) + 72z^2(\theta+1)^2;$

(h) $A_n = 27^n \sum_k (-1)^k \binom{-2/3}{k}\binom{-1/3}{n-k}^2,$

$$D = \theta^2 - 3z(18\theta^2 + 18\theta + 7) + 729z^2(\theta+1)^2;$$

(i) $A_n = 64^n \sum_k (-1)^k \binom{-3/4}{k}\binom{-1/4}{n-k}^2,$

$$D = \theta^2 - 4z(32\theta^2 + 32\theta + 13) + 4096z^2(\theta+1)^2;$$

(j) $A_n = 432^n \sum_k (-1)^k \binom{-5/6}{k}\binom{-1/6}{n-k}^2,$

$$D = \theta^2 - 12z(72\theta^2 + 72\theta + 31) + 186624z^2(\theta+1)^2.$$

In case (g), no explicit formulas for A_n is known. In other cases the explicit formulas for A_n were found in the following way.

The Legendre function

$$P_a(z) = {}_2F_1\left(\begin{matrix} -a, & a+1 \\ & 1 \end{matrix} \,\middle|\, \frac{1-z}{2} \right)$$

satisfies the differential equation

$$(1-z^2)\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}y}{\mathrm{d}z} + a(a+1)y = 0.$$

Therefore, the function

$$y_0(z) = \frac{1}{1-z} P_a\left(\frac{1+z}{1-z}\right)$$

is annihilated by the differential operator

$$\theta^2 - z(2\theta^2 + 2\theta + a^2 + a + 1) + z^2(\theta+1)^2$$

(see [Za]). We have

$$(1-z)^a P_a\left(\frac{1+z}{1-z}\right) = \sum_{n=0}^{\infty} \binom{a}{n}^2 z^n$$

which gives

$$\frac{1}{1-cz} P_a\left(\frac{1+cz}{1-cz}\right) = \sum_{n=0}^{\infty} A_n z^n,$$

where

$$A_n = c^n \sum_k (-1)^k \binom{-1-a}{k}\binom{a}{n-k}^2.$$

Putting $a = -1/3, -1/4, -1/6$ for a suitable choice of c we get cases (h)–(j).

Let further

$$\text{(A)} \qquad A_n = \binom{2n}{n}^2, \qquad D = \theta^2 - 4z(2\theta+1)^2;$$

$$\text{(B)} \qquad A_n = \frac{(3n)!}{n!^3}, \qquad D = \theta^2 - 3z(3\theta+1)(3\theta+2);$$

$$\text{(C)} \qquad A_n = \frac{(4n)!}{n!^2(2n)!}, \qquad D = \theta^2 - 4z(4\theta+1)(4\theta+3);$$

$$\text{(D)} \qquad A_n = \frac{(6n)!}{n!(2n)!(3n)!}, \qquad D = \theta^2 - 12z(6\theta+1)(6\theta+5).$$

One can prove the following formula for the Hadamard product by the method presented below (P and Q are quadratic polynomials and c is an arbitrary constant):

$$\big(\theta^2 - zP(\theta) - cz^2(\theta+1)^2\big) * \big(\theta^2 - zQ(\theta)\big) = \theta^4 - zP(\theta)Q(\theta) - cz^2 Q(\theta+1)Q(\theta).$$

Then we get the following table of cases[3] from [AESZ, Table A] corresponding to Hadamard products:

[3]We will not write out in [AESZ, Table A] the equations in cases #133–143, since the reader can easily derive them himself by the formula above. We have, however, checked that one gets integers everywhere.

	(A)	(B)	(C)	(D)
(a)	45	15	68	62
(b)	25	24	51	63
(c)	58	70	69	64
(d)	36	48	38	65
(e)	111	110	30	112
(f)	133	134	135	136
(g)	137	138	139	140
(h)	141	142	$\emptyset$	143

The empty set at place (C)$*$(h) means that the Yukawa coupling $K(q)$ is constant.

In our computations we applied the following method communicated to us by A. Meurman. The method is, in a sense, a discrete analogue of the wronskian formalism.

Method to find Hadamard products. Assume

$$A_{n+2} = P_1(n)A_{n+1} + Q_1(n)A_n, \quad B_{n+2} = P_2(n)B_{n+1} + Q_2(n)B_n,$$

and we want to find a recursion formula for $C_n = A_nB_n$. For this, define

$$R_j(n) = P_j(n+1)P_j(n) + Q_j(n+1), \quad S_j(n) = P_j(n+1)Q_j(n), \qquad j = 1, 2,$$

$$\begin{aligned} U_j(n) &= P_j(n+2)R_j(n) + Q_j(n+2)P_j(n), \\ V_j(n) &= P_j(n+2)S_j(n) + Q_j(n+2)Q_j(n), \end{aligned} \qquad j = 1, 2,$$

and take

$$T_2(n) = R_2S_1U_1V_2 - R_1S_2U_2V_1, \quad T_3(n) = P_1Q_2U_2V_1 - P_2Q_1U_1V_2,$$
$$T_4(n) = R_1S_2P_2Q_1 - R_2S_1P_1Q_2,$$

and

$$W_0(n) = Q_1Q_2T_2 + S_1S_2T_3 + V_1V_2T_4, \quad W_1(n) = P_1P_2T_2 + R_1R_2T_3 + U_1U_2T_4.$$

Then

$$T_4(n)C_{n+4} + T_3(n)C_{n+3} + T_2(n)C_{n+2} - W_1(n)C_{n+1} - W_0(n)C_n = 0.$$

When $A_n = B_n$, the method has to be modified. Assume

$$A_{n+2} = P(n)A_{n+1} + Q(n)A_n,$$

where $P(n)$ and $Q(n)$ are rational. Then

$$A_{n+3} = R(n)A_{n+1} + S(n)A_n,$$

where

$$R(n) = P(n+1)P(n) + Q(n+1), \quad S(n) = P(n+1)Q(n).$$

Letting $C_n = A_n^2$ and $T(n) = R(n)Q(n) - P(n)S(n)$, we see that

$$P(n)Q(n)C_{n+3} - R(n)S(n)C_{n+2} - P(n)R(n)T(n)C_{n+1} + Q(n)S(n)T(n)C_n = 0.$$

The procedure leads usually (in cases (a)–(h)) to a 8th order differential equation (6th order if $A_n = B_n$) that factors into a 4th order equation. Cases #167–179 were computed by this way.

Actually we found four 2nd order differential equations somewhat suitable for Hadamard products:

(k) $A_n = 9^n \sum_k (-1)^k \binom{-1/3}{k} \binom{-2/3}{n-k} \binom{n}{k}$, $D = \theta^2 - 3z(2\theta+1) - 81z^2(\theta+1)^2$;

(l) $A_n = 8^n \sum_k (-1)^k \binom{-1/4}{k} \binom{-3/4}{n-k} \binom{n}{k}$, $D = \theta^2 - 4z(2\theta+1) - 64z^2(\theta+1)^2$;

(m) $A_n = 36^n \sum_k (-1)^k \binom{-1/6}{k} \binom{-5/6}{n-k} \binom{n}{k}$,
$D = \theta^2 - 24z(2\theta+1) - 1296z^2(\theta+1)^2$;

(n) $A_n = 4^n \sum_k (-1)^k \binom{-1/2}{k} \binom{-1/2}{n-k} \binom{n}{k}$, $D = \theta^2 - 16z^2(\theta+1)^2$.

Only (n) gives Calabi-Yau differential equations when we form (a) $*$ (n), (b) $*$ (n), …, (n) $*$ (n) but this is just the substitution $z \mapsto z^{1/2}$. The Hadamard squares of (k), (l), (m) correspond to Calabi–Yau differential equations found from the formula

$$\begin{aligned}
&(\theta^2 - Az(2\theta+1) - Bz^2(\theta+1)^2)^{*2} \\
&\quad = \theta^4 - z(2B\theta^4 + A^2(2\theta+1)^2) - Bz^2(2\theta+1)(2B\theta^2 + 2B\theta + 4A^2 + B) \\
&\qquad + B^2z^3(2B\theta^4 + 8B\theta^3 + (4A^2 + 12B)\theta^2 + (4A^2 + B)\theta + A^2 + 2B) \\
&\qquad - B^4z^4(\theta+1)^4.
\end{aligned}$$

As a curiosity we mention that in the case (m) $*$ (m) we have

$$q = z(1 + 21z + 3840z^2 + \cdots)^{12960},$$

where the series in the brackets is expected to lie in $\mathbb{Z}[[z]]$.

Of all non-square Hadamard products only (k) $*$ (m) gives a Calabi–Yau equation

$$\begin{aligned}
&\theta^4 - 4\cdot 18z(2\theta+1)^2 - 2\cdot 18^3z^2(27\theta^4 + 36\theta^3 + 74\theta^2 + 76\theta + 24) \\
&\quad - 32\cdot 18^5z^3(2\theta+1) + 2\cdot 18^7z^4(27\theta^4 + 72\theta^3 + 128\theta^2 + 72\theta + 13) \\
&\quad + 4\cdot 18^9z^5(2\theta+1)^2 - 18^{12}z^6(\theta+1)^4.
\end{aligned}$$

To get integer instanton numbers we need to make the substitution

$$\widetilde{K}(q) = K(q/3) = 1 + \sum_{k=1}^{\infty} \frac{k^3 \widetilde{N}_k q^k}{1-q^k};$$

then

$$\widetilde{N}_1 = -48, \quad \widetilde{N}_2 = -126, \quad \widetilde{N}_3 = -2864, \quad \widetilde{N}_4 = 77958, \quad \widetilde{N}_5 = 4942032, \quad \dots .$$

We can also get 5th order equations by taking the Hadamard product of (A)–(D) with 3rd order equations of the following type:

$$(\alpha) \quad A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2n-2k}{n-k},$$
$$D = \theta^3 - 2z(2\theta+1)(5\theta^2+5\theta+2) + 64z^2(\theta+1)^3;$$

$$(\beta) \quad A_n = \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2,$$
$$D = \theta^3 - 8z(2\theta+1)(2\theta^2+2\theta+1) + 256z^2(\theta+1)^3;$$

$$(\gamma) \quad A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$
$$D = \theta^3 - z(2\theta+1)(17\theta^2+17\theta+5) + z^2(\theta+1)^3;$$

$$(\delta) \quad A_n = \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \binom{n+k}{k} \frac{(3k)!}{k!^3},$$
$$D = \theta^3 - z(2\theta+1)(7\theta^2+7\theta+3) + 81z^2(\theta+1)^3;$$

$$(\epsilon) \quad A_n = \sum_{k} \binom{n}{k}^2 \binom{2k}{n}^2,$$
$$D = \theta^3 - 4z(2\theta+1)(3\theta^2+3\theta+1) + 16z^2(\theta+1)^3;$$

$$(\zeta) \quad D = \theta^3 - 3z(2\theta+1)(3\theta^2+3\theta+1) - 27z^2(\theta+1)^3;$$

$$(\eta) \quad D = \theta^3 - z(2\theta+1)(11\theta^2+11\theta+5) + 125z^2(\theta+1)^3,$$

$$(\vartheta) \quad A_n = (-64)^n \sum_{k} \binom{-1/2}{k} \binom{-1/2}{n-k}^3 = 64^n \sum_{k} \binom{-1/4}{k}^2 \binom{-3/4}{n-k}^2,$$
$$D = \theta^3 - 8z(2\theta+1)(8\theta^2+8\theta+5) + 4096z^2(\theta+1)^3,$$

$$(\iota) \quad A_n = 27^n \sum_{k} \binom{-1/3}{k}^2 \binom{-2/3}{n-k}^2,$$
$$D = \theta^3 - 3z(2\theta+1)(9\theta^2+9\theta+5) + 729z^2(\theta+1)^3,$$

$$(\kappa) \quad A_n = 432^n \sum_{k} \binom{-1/6}{k}^2 \binom{-5/6}{n-k}^2,$$

$$D = \theta^3 - 24z(2\theta+1)(18\theta^2+18\theta+13) + 186624z^2(\theta+1)^3.$$

The last 6 examples were found in [ES] (without formulas for A_n).

The general formula for the Hadamard product in this case is as follows:

$$\begin{aligned}&\big(\theta^2 - z(2\theta+1)P(\theta) - cz^2(\theta+1)^3\big) * \big(\theta^2 - zQ(\theta)\big)\\ &\quad= \theta^5 - z(2\theta+1)P(\theta)Q(\theta) - cz^2(\theta+1)Q(\theta+1)Q(\theta)\end{aligned}$$

and we obtain the following table of corresponding cases[4]:

	(A)	(B)	(C)	(D)
(α)	39	61	37	66
(β)	40	49	43	67
(γ)	44	53	52	149
(δ)	150	151	152	153

In addition, we have another type of Hadamard products. If $\{A_n\}$ is given by any of the cases (α)–(ϵ) with equation

$$\theta^3 - z(2\theta+1)P(\theta) + cz^2(\theta+1)^3,$$

then the generating series for the sequence $\{\binom{2n}{n}A_n\}$ is annihilated by the 4th-order differential equation

$$\theta^4 - 2z(2\theta+1)^2P(\theta) + 4cz^2(\theta+1)^2(2\theta+1)(2\theta+3).$$

Thus we get cases #16, 35, 29, 41, and 42, respectively.

Proposition 9. *Any differential equation of type $D'y=0$, where*

$$D' = \theta^3 - z(2\theta+1)(a\theta^2+a\theta+b) + cz^2(\theta+1)^3,$$

is the symmetric square of $Dy=0$, where

$$D = \theta^2 - z(2a\theta^2 + a\theta + \tfrac{b}{2}) + cz^2(\theta+\tfrac12)^2. \tag{7.2}$$

Proof. In [Al2] it is proved that the differential equation

$$y^{(3)} + s_2(z)y'' + s_1(z)y' + s_0(z)y = 0 \tag{7.3}$$

is a symmetric square of

$$u'' + p_1(z)u' + p_0(z)u = 0 \tag{7.4}$$

[4]We computed all the other pullbacks of the Hadamard products (one of those has trivial $K(q)$) and found no new instanton numbers: they are all identical with earlier known Hadamard products of type $(a)*(b)$ etc. The pullbacks are contained in the database [En].

if and only if

$$\frac{s_1 s_2}{3} - \frac{2s_2^3}{27} + \frac{s_1'}{2} - \frac{s_2''}{6} - \frac{s_2 s_2'}{3} - s_0 = 0,$$

and then

$$p_1 = \frac{s_2}{3}, \qquad p_0 = \frac{s_1}{4} - \frac{s_2^2}{18} - \frac{s_2'}{12}.$$

Here we get

$$s_0 = \frac{-b + cz}{1 - 2az + cz^2}, \qquad s_1 = \frac{1 - (6a + 2b)z + 7cz^2}{1 - 2az + cz^2}, \qquad s_2 = \frac{3(1 - 3az + 2cz^2)}{1 - 2az + cz^2},$$

which satisfy the identity above. We also get

$$p_1 = \frac{1 - 3az + 2cz^2}{z(1 - 2az + cz^2)}, \qquad p_0 = \frac{-2b + cz}{4z(1 - 2az + cz^2)},$$

and the operator $\mathrm{d}^2/\mathrm{d}z^2 + p_1(z)\,\mathrm{d}/\mathrm{d}z + p_0(z)$ is easily converted to (7.2). $\square$

Remark. The Frobenius basis of the 3rd order differential equation (7.3) is

$$y_0 = u_0^2, \quad y_1 = u_0 u_1, \quad y_2 = \frac{1}{2} u_1^2,$$

where u_0, u_1 form the Frobenius basis of (7.4). Hence

$$\frac{y_2}{y_0} = \frac{1}{2}\left(\frac{u_1}{u_0}\right)^2 = \frac{1}{2}\left(\frac{y_1}{y_0}\right)^2 = \frac{t^2}{2}$$

and

$$K(q) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{y_2}{y_0}\right) = 1 = \text{constant}.$$

The substitution $z \mapsto 4z$ makes also $u_0 = y_0^{1/2}$ to have integer coefficients. Since θ is invariant under this substitution we get

$$D \mapsto \theta^2 - z(8a\theta^2 + 4a\theta + 2b) + 4cz^2(2\theta + 1)^2.$$

Finding such equations is equivalent to finding integer solutions to the recursion

$$(n+1)^2 u_{n+1} = 2(a \cdot 2n(2n+1) + 2b)u_n - 4c(2n-1)^2 u_{n-1}$$

which is similar to Zagier's computation in [Za].

On the other hand, we found that only 2nd-order differential operators of type $\theta^2 - zP(\theta) + cz^2(\theta + 1)^2$, where $P(\theta)$ is a polynomial of degree 2, were useful for producing 4th-order Calabi–Yau by taking the Hadamard product of two of them. This, however, coincides with Zagier's list [Za] of integer solutions to

$$(n+1)^2 A_{n+1} - an(n+1)A_n + bn^2 A_{n-1} = \lambda A_n.$$

Zagier's list has 36 entries. It contains our cases (a)–(h), (A)–(C) (but not (D) since the coefficients are too large). If A_n is a polynomial in n (that corresponds to 8 entries in the list), then the corresponding 2nd-order differential equation factors. Excluding these polynomial cases, hypergeometric cases (A)–(C) and terminating cases, we found that certain remaining cases contain examples that we do not really like. For instance, the recurrence

$$(n+1)^2A_{n+1} - 32n(n+1)A_n + 256A_{n-1} = 28A_n$$

admits the following integer-valued solution:

$$A_n = 2^{4n}\sum_{k=0}^{n}\binom{1/2}{n-k}^2\binom{k+1/2}{k} = \sum_{k=0}^{n}2^{2k}\frac{2k+1}{(2n-2k+1)^2}\binom{2n-2k}{n-k}^2\binom{2k}{k};$$

the 4th-order differential operator annihilating the series $\sum_{n=0}^{\infty}\binom{2n}{n}^2A_nz^n$ is as follows:

$$\theta^4 - 16z(2\theta+1)^2(8\theta^2+8\theta+7) + 4096z^2(2\theta+1)^2(2\theta+3)^2$$

and the corresponding differential equation satisfies relation (2.2); then computations show that $z(q) \notin \mathbb{Z}[[q]]$ (and N_l in (2.9) are not integers). This is an example promised in Remark of Section 5.

8. More transformations

Transformations of the first 14 *hypergeometric cases.* The success with negative rational numbers in binomial coefficients caused us to make the following experiment. Taking the hypergeometric operator in case #9,

$$\theta^4 - 12^6z\left(\theta+\frac{1}{12}\right)\left(\theta+\frac{5}{12}\right)\left(\theta+\frac{7}{12}\right)\left(\theta+\frac{11}{12}\right),$$

consider

$$A_n = 1728^n\binom{2n}{n}\sum_k\binom{-1/12}{k}\binom{-5/12}{k}\binom{-7/12}{n-k}\binom{-11/12}{n-k}, \qquad n=0,1,2,\dots.$$

Then the series $y_0(z) = \sum_{n=0}^{\infty}A_nz^n$ satisfies the differential equation $Dy=0$, where

$$D = \theta^4 - 48z(2\theta+1)^2(72\theta^2+72\theta+41) + 2^{14}\cdot 3^4z^2(2\theta+1)(2\theta+3)(3\theta+2)(3\theta+4),$$

an equation found in [ES] and mentioned as case #9* in [AESZ, Table A]. We get the instanton numbers

$$N_1 = -480, \quad N_2 = -226848, \quad N_3 = 16034720,$$
$$N_4 = 1094330202744, \quad N_5 = 4352645747040, \quad \dots.$$

We also note that

$$y_0 = (1+246z+768132z^2+\cdots)^8, \qquad q = z(1+12z+33333z^2+\cdots)^{288},$$

where the series in brackets are expectively in $\mathbb{Z}[[z]]$. Interchanging 5/12 and 7/12, we obtain case 9**:

$$\begin{aligned} A_n &= 1728^n \binom{2n}{n} \sum_k \binom{-1/12}{k}\binom{-7/12}{k}\binom{-5/12}{n-k}\binom{-11/12}{n-k} \\ &= 432^n \binom{2n}{n}^2 \sum_k (-1)^k \binom{-5/6}{k}\binom{-1/6}{n-k}^2, \end{aligned}$$

with the corresponding differential operator

$$D = \theta^4 - 48z(2\theta+1)^2(72\theta^2+72\theta+31) + 2^{12}\cdot 3^6 z^2(2\theta+1)^2(2\theta+3)^2;$$

the coupling $K(q)$ is the same as in #9*.

Trying the above experiment for all equations #1–#14 we find that cases #1, #2, #11, #12 and #14 give integer mirror map and Yukawa coupling but *there seems to be no way to get integer instanton numbers* (by doing the substitution $z \to cz$). Case #5* has trivial $K(q)$. We have, however, a couple of 'good' cases #3*, #4*, #4**, #6*, #7*, #7**, #8*, #8**, #10*, #10**, #13* and #13** (see Appendix A). This (together with cases #9*, #9** given above) ends the list of the 14 new equations.

By some unusual substitions we can also consider the following additional cases. In case #14*, we take

$$A_n = 144^n \binom{2n}{n} \sum_k \binom{-1/6}{k}\binom{-1/2}{k}\binom{-5/6}{n-k}\binom{-1/2}{n-k}$$

and obtain the corresponding differential operator

$$D = \theta^4 - 2^4\cdot 3^2 z(2\theta+1)^2(2\theta^2+2\theta+1) + 2^{10}\cdot 3^2 z^2(2\theta+1)(2\theta+3)(3\theta+2)(3\theta+4).$$

Then, choosing

$$\widetilde{K}(q) = K\big((q/3)^{1/2}\big) = 1 + \sum_{k=1}^{\infty} \frac{k^3 \widetilde{N}_k q^k}{1-q^k},$$

we have

$$\widetilde{N}_1 = -2592, \quad \widetilde{N}_2 = -307800, \quad \widetilde{N}_3 = 81451104,$$
$$\widetilde{N}_4 = 144135316512, \quad \widetilde{N}_5 = 98667659422368, \quad \dots .$$

In case #2*, when

$$A_n = 2000^n \binom{2n}{n} \sum_k \binom{-1/10}{k}\binom{-3/10}{k}\binom{-7/10}{n-k}\binom{-9/10}{n-k},$$
$$\begin{aligned} D = \theta^4 &- 2^4\cdot 5z(2\theta+1)^2(50\theta^2+50\theta+33) \\ &+ 2^{10}\cdot 5^4 z^2(2\theta+1)(2\theta+3)(5\theta+4)(5\theta+6), \end{aligned}$$

we need even more drastic methods; the 'instanton numbers' land in $\mathbb{Z}[\sqrt{5}]$. Taking

$$\widetilde{K}(q) = K(q/\sqrt{5}) = 1 + \sum_{k=1}^{\infty} \frac{k^3 \widetilde{N}_k q^k}{1-q^k},$$

we obtain

$$\widetilde{N}_1 = -256\sqrt{5}, \quad \widetilde{N}_2 = 32\sqrt{5} - 35260, \quad \widetilde{N}_3 = 1004288\sqrt{5},$$
$$\widetilde{N}_4 = 835297220, \quad \widetilde{N}_5 = 102454248704\sqrt{5}, \quad \dots .$$

Case #2** corresponds to the interchange of 3/10 and 7/10, hence the change of the factor $50\theta^2 + 50\theta + 33$ by $50\theta^2 + 50\theta + 17$ in the differential operator and conjugate $\widetilde{K}(q)$. Is there a geometric interpretation of the numbers $\widetilde{N}_k$?

Differential equations inspired by Guillera's formulas. The formulas discovered in [Gu] caused us to try the following (again we mimic case #9). Let

$$A'_n = 3456^{2n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(1/2)_k (1/12)_k (5/12)_k (7/12)_k (11/12)_k}{k!^5}.$$

Then $w_0 = \sum_{n=0}^{\infty} A'_n z^n$ satisfies $D'w = 0$, where

$$\begin{aligned} D' = \theta^5 &- 288z(2\theta+1)(103680\theta^4 + 207360\theta^3 + 262944\theta^2 + 159264\theta + 41087) \\ &+ 2^{20}3^8 z^2(\theta+1)(207360\theta^4 + 829440\theta^3 + 1514592\theta^2 + 1370304\theta + 498143) \\ &- 2^{38}3^{17} z^3(\theta+1)(\theta+2)(2\theta+3)(240\theta^2 + 720\theta + 793) \\ &+ 2^{53}3^{22} z^4(\theta+1)(\theta+2)(\theta+3)(360\theta^2 + 1440\theta + 1633) \\ &- 2^{69}3^{30} z^5(\theta+1)(\theta+2)(\theta+3)(\theta+4)(2\theta+5). \end{aligned}$$

The 4th order pullback of the latter differential operator is

$$\begin{aligned} D = \theta^4 &- 2^4 3^2 z(248832\theta^4 + 414720\theta^3 + 318528\theta^2 + 111168\theta + 14497) \\ &+ 2^{22}3^{10} z^2(4\theta+3)(432\theta^3 + 1116\theta^2 + 886\theta + 165) \\ &- 2^{34}3^{18} z^3(4\theta+1)(4\theta+3)(4\theta+7)(4\theta+9) \end{aligned}$$

and, remarkably, it has lower degree than the 5th order equation (cf. case #129). We get the instanton numbers

$$N_1 = 2710944, \quad N_2 = -717640978896, \quad N_3 = 302270555492914464,$$
$$N_4 = -171507700573958028578832, \quad N_5 = 113303073680022744870130144224, \quad \dots,$$

and note that

$$y_0 = (1 + 260946z + 1405445560884z^2 + \cdots)^8,$$
$$q = x(1 + 13295z + 67507583411z^2 + \cdots)^{576}, \quad K(q) = (1 + 56478q - \cdots)^{48},$$

where the parentheses are supposed to have integer coefficients.

Doing the same thing with case #3, i.e. taking

$$A'_n = 1024^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(1/2)_k^5}{k!^5},$$

we get the same Yukawa coupling as for case #115 which is an Hadamard square with

$$A_n = \left\{ \sum_k 4^{n-k} \binom{2k}{k}^2 \binom{2n-2k}{n-k}^2 \right\}^2$$

But the differential equations are quite different as are the solutions y_0.

We list all possible variations on the theme as cases $\#\widehat{1}$–$\widehat{14}$ in [AESZ, Table A].

There is an even simpler way to arrive at essentially the same Yukawa coupling $K(q)$ as above. Consider the 5th order hypergeometric operator

$$\theta^5 - 4 \cdot 12^6 z\Big(\theta + \frac{1}{2}\Big)\Big(\theta + \frac{1}{12}\Big)\Big(\theta + \frac{5}{12}\Big)\Big(\theta + \frac{7}{12}\Big)\Big(\theta + \frac{11}{12}\Big),$$

with the solution

$$w_0 = \sum_{n=0}^{\infty} \frac{(12n)!}{(6n)!n!^6} \frac{z^n}{\binom{4n}{2n}}$$

of the corresponding differential equation $D'w = 0$, and with the 4th order pullback

$$\begin{aligned} D = \theta^4 &- 2^4 3^2 z(3317760\theta^4 + 829440\theta^3 + 132480\theta^2 - 282240\theta - 14497) \\ &+ 2^{19} 3^8 z^2 (2488320\theta^4 + 1244160\theta^3 + 250560\theta^2 - 4176\theta + 21143) \\ &- 2^{32} 3^{14} z^3 (3317760\theta^4 + 2488320\theta^3 + 604800\theta^2 + 213120\theta - 4453) \\ &+ 2^{51} 3^{22} z^4 \theta(2\theta + 1)(12\theta + 1)(12\theta + 5) \end{aligned}$$

Observe that here the pullback increased the degree from 1 to 4. The instanton numbers are

$$N_1 = -2710944, \quad N_2 = -717640301160, \quad \dots,$$

i.e. the same as above up to a sign. We also note

$$y_0 = (1 - 86982z - 577991455848z^2 - \cdots)^{24}, \quad q = x(1 + 7441z + 32486988467z^2 + \cdots)^{576}.$$

A third variation of the theme is the following. Comsider

$$A'_n = (4 \cdot 12^6)^n \binom{-1/2}{n} \sum_{k=0}^{n} \binom{n}{k} \binom{-1/12}{k} \binom{-5/12}{k} \binom{-7/12}{k} \binom{-11/12}{k}$$

with corresponding 5th order differential operator

$$\begin{aligned} D' = \theta^5 &+ 2 \cdot 12^2 z(2\theta + 1)(1244160\theta^4 + 2488320\theta^3 + 2347200\theta^2 + 1103040\theta + 21121) \\ &+ 112 \cdot 12^{10} z^2 (\theta + 1)(2\theta + 1)(2\theta + 3)(720\theta^2 + 1440\theta + 101) \\ &+ 16 \cdot 12^{16} z^3 (2\theta + 1)(2\theta + 3)(2 + 5)(1152\theta^2 + 3456\theta + 2831) \\ &+ 12^{26} z^4 (\theta + 2)(2\theta + 1)(2\theta + 3)(2\theta + 5)(2\theta + 7) \\ &+ 64 \cdot 12^{30} z^5 (2\theta + 1)(2\theta + 3)(2\theta + 5)(2 + 7)(2\theta + 9) \end{aligned}$$

with pullback

$$\begin{aligned}
D = \theta^4 &+ 2^4 3^2 z(995328\theta^4 + 497664\theta^3 + 220608\theta^2 - 28224\theta - 35233)\\
&+ 2^{18}3^8 z^2(5142528\theta^4 + 5308416\theta^3 + 2946816\theta^2 + 9792\theta - 292949)\\
&+ 2^{35}3^{14} z^3(1866240\theta^4 + 2985984\theta^3 + 2011320\theta^2 + 142920\theta - 208501)\\
&+ 2^{48}3^{20} z^4(66562560\theta^4 + 146810880\theta^3 + 117328320\theta^2 + 13269600\theta - 1309273)\\
&+ 2^{64}3^{28} z^5(4\theta-1)(518400\theta^3 + 1607040\theta^2 + 1779080\theta + 64537)\\
&+ 2^{79}3^{34} z^6(4\theta-1)(4\theta+3)(44640\theta^2 + 135360\theta + 10985)\\
&+ 2^{97}3^{43} z^7(\theta+2)(4\theta-1)(4\theta+3)(4\theta+7)\\
&+ 2^{108}3^{48} z^8(4\theta-1)(4\theta+3)(4\theta+7)(4\theta+11).
\end{aligned}$$

We have

$$y_0 = (1 + 211398z - 2341345644648z^2 + \cdots)^{24}$$

and

$$q = z(1 - 28177z + 245565832115z^2 - \cdots)^{576}.$$

The instanton numbers are the same as above. Thus we have three fourth order differential equations of degree 3, 4, and 8, respectively. The instanton numbers, however, are invariants (possibly, with some geometric interpretation). It would be interesting to find the transformations between the three different solutions.

The mirror at infinity. In [Ro] E. Rødland studied case #27 at infinity. We choose instead #124 from [ES]:

$$\begin{aligned}
D = 61^2\theta^4 &- 61z(3029\theta^4 + 5572\theta^3 + 4677\theta^2 + 1891\theta + 305)\\
&+ z^2(121521\theta^4 + 342813\theta^3 + 426722\theta^2 + 257267\theta + 61158)\\
&- 3^4 z^3(39370\theta^4 + 140178\theta^3 + 206807\theta^2 + 142191\theta + 37332)\\
&+ 3^8 z^4(566\theta^4 + 2230\theta^3 + 3356\theta^2 + 2241\theta + 558) - 3^{13} z^5(\theta+1)^4,
\end{aligned}$$

where the explicit formulas for A_n are known and one should take $N_0 = 61$ in order to get integer instanton numbers (cf. (2.9)). The substitution $z \mapsto 3^{-5}z^{-1}$, $y \mapsto zy$ results in $\theta \to -\theta - 1$ (this works only if the highest degree term has the form $(\theta+1)^4$) and we obtain the 'dual' differential operator

$$\begin{aligned}
D^* = \theta^4 &- z(566\theta^4 + 34\theta^3 + 62\theta^2 + 45\theta + 9)\\
&+ 3z^2(39370\theta^4 + 17302\theta^3 + 22493\theta^2 + 8369\theta + 1140)\\
&- 3^2 z^3(121521\theta^4 + 143272\theta^3 + 127412\theta^2 + 53824\theta + 9322)\\
&+ 3^7 61 z^4(3029\theta^4 + 6544\theta^3 + 6135\theta^2 + 2863\theta^2 + 548) - 3^{12}61^2 z^5(\theta+1)^4.
\end{aligned}$$

We do not know a formula for the corresponding A_n here.

Many differential equations with highest degree term $(\theta+1)^4$ (in particular, all Hadamard products) are self-dual. There is, however, the class with

$$A_n = \sum_k \binom{n}{k}^{5-2r} \binom{2k}{k}^r \binom{2n-2k}{n-k}^r$$

for $r = 0, 1, 2, 3, 4, 5$ (cases #22, #21, #23, #56, #71, #118, respectively). Then the duality maps $r \leftrightarrow 5 - r$.

Also differential equations with highest degree term $(2\theta+1)^4$ can be reflected in ∞. Let us take, e.g., case #55 with

$$\begin{aligned} D = 9\theta^4 &- 12z(208\theta^4 + 224\theta^3 + 163\theta^2 + 51\theta + 6) \\ &+ 2^9 z^2(32\theta^4 - 928\theta^3 - 1606\theta^2 - 837\theta - 141) \\ &+ 2^{16} z^3(144\theta^2 + 576\theta^3 + 467\theta^2 + 144\theta + 15) - 2^{24} z^4(2\theta + 1)^4. \end{aligned}$$

Then the transformation $z \to 2^{-18} z^{-1}$ maps θ to $-\theta - 1/2$ and gives the dual

$$\begin{aligned} D^* = \theta^4 &- 2^4 z(576\theta^4 - 1152\theta^3 - 724\theta^2 - 148\theta - 13) \\ &- 2^{17} z^2(32\theta^4 + 992\theta^3 - 166\theta^2 - 57\theta - 6) \\ &+ 2^{26} 3 z^3(832\theta^4 + 768\theta^3 + 556\theta^2 + 192\theta + 25) - 2^{40} 3^2 z^4(2\theta + 1)^4 \end{aligned}$$

Also cases #33, #99, #154 can be treated in the same way (case #154 is self-dual).

9. Supercongruences and k-realizable series

It is time to stop and discuss the phenomenon of integer coefficients of Yukawa couplings number-theoretically.

Let $T\colon X \to X$ be a map of a set. Define the number of points x with period n:

$$f_n(T) = \#\{x \in X : T^n x = x\},$$

and the number of points x with the smallest period n:

$$f_n^*(T) = \#\big\{x \in X : T^n x = x,\ \#\{T^l x : l \in \mathbb{Z}\} = n\big\}.$$

Then $f_n^0(T) = \frac{1}{n} f_n^*(T)$ determines the number of orbits of length n. Applying the Möbius inversion formula to $f_n(T) = \sum_{d|n} f_d^*(T)$ we obtain

$$f_n^0(T) = \frac{1}{n} \sum_{d|n} \mu\Big(\frac{n}{d}\Big) f_d(T). \tag{9.1}$$

Proposition 10 [PW]. *Let $\{A_n\}_{n=1}^{\infty}$ be a sequence with all $a_n \geqslant 0$. Then, there exists a map $T\colon X \to X$ such that $A_n = f_n(T)$ if and only if the numbers*

$$B_n = \frac{1}{n} \sum_{d|n} \mu\Big(\frac{n}{d}\Big) A_d, \qquad n = 1, 2, \dots,$$

are all non-negative integers.

Proof. The '$\Longrightarrow$'-part was already done in (9.1).

In order to prove the '$\Longleftarrow$'-part, we take $X = \{0, 1, 2, \dots\}$ and let $T\colon X \to X$ be a permutation consisting of B_n cycles of length n for all $n = 1, 2, \dots$. Then $A_n = f_n(T)$ as required. □

To illustrate the above construction of T in the proof, we consider the following

Example. Let $\{A_n\}_{n=1}^{\infty} = \{1, 5, 7, 17, 31, 65, \dots\}$ be given by the recurrence $A_{n+1} = A_n + 2A_{n-1}$ for $n = 1, 2, \dots$. Then $\{B_n\}_{n=1}^{\infty} = \{1, 2, 2, 3, 6, 9, \dots\}$ and a choice of T is as follows:

$$T = \underbrace{(0)}_{B_1=1}\ \underbrace{(1\ 2)(3\ 4)}_{B_2=2}\ \underbrace{(5\ 6\ 7)(8\ 9\ 10)}_{B_3=2}\ \underbrace{(11\ 12\ 13\ 14)(15\ 16\ 17\ 18)(19\ 20\ 21\ 22)}_{B_4=3}\dots.$$

The points of order 4 are

$$\{11, 12, \dots, 22,\ 1, 2, 3, 4,\ 0\},$$

hence $f_4(T) = 4 \cdot 3 + 2 \cdot 2 + 1 = 17 = A_4$.

Assume now that for some $k \geqslant 1$ we have

$$\frac{1}{n^k} \sum_{d|n} \mu\Big(\frac{n}{d}\Big) A_d \in \mathbb{N} \qquad \text{for all } n = 1, 2, \dots.$$

This means that the number of orbits of length n is divisible by n^{k-1}. The sequence $\{A_n\}_{n=1,2,\dots}$ is said to be *k-realizable* if

$$B_n = \frac{1}{n^k} \sum_{d|n} \mu\Big(\frac{n}{d}\Big) A_d \in \mathbb{Z} \qquad \text{for all } n = 1, 2, \dots \tag{9.2}$$

holds. Call a power series $y = \sum_{n=0}^{\infty} A_n z^n \in \mathbb{Q}[[z]]$ *k-realizable* if there exists an integer C such that $\{CA_n\}_{n=1}^{\infty}$ is k-realizable.

The most remarkable property of the examples of the Calabi–Yau differential equations is that the Yukawa coupling (2.9) is (expected to be) 3-realizable; the pseudo-coupling (3.8) constructed in Section 3 seems to be 2-realizable (where in cases #15–29 we have to replace $\mathbb{Z}$ in (9.2) by $\mathbb{Z}[1/p]$ for a suitable choice of the prime p). Another striking property of the first 14 (hypergeometric) cases from [AESZ, Table A] is that $y_0(z)$ is also 3-realizable.

The property of being k-realizable is related to Kummer supercongruences by the following result.

Proposition 11. *The following are equivalent:*

(i) $\{A_n\}$ *is k-realizable;*
(ii) $A_{mp^r} \equiv A_{mp^{r-1}} \pmod{p^{rk}}$ *for p prime and* $(m, p) = 1$*;*
(iii) $\sum_{n=1}^{\infty} A_n z^n / n^k = \sum_{n=1}^{\infty} B_n \operatorname{Li}_k(z^n)$, *where B_n are the integers defined in* (9.2) *and*

$$\operatorname{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$$

denotes the k-th polylogarithm.

Proof. The proofs on pp. 3–6 in [Re] for $k = 1$ are easily generalized; note that $\operatorname{Li}_1(x) = -\log(1 - x)$. □

Remark. For the first n numbers $A_1, A_2, \dots, A_n$, the case $k = 1$ is well described in Exercise 5.2 of [St]. Then there is also an $(n \times n)$-matrix M with integer entries such that $A_k = \operatorname{Tr}(M^k)$ for $k = 1, 2, \dots, n$ (see [Al1]).

10. 6th- and higher order differential equations

We would like to start this last section by presenting two other puzzling examples related to 4-term polynomial recursions

$$\begin{aligned}
&(n+1)^6(41218n^3-48459n^2+20010n-2871)A_{n+1}\\
&\quad+2(48802112n^9+89030880n^8+36002654n^7-24317344n^6-19538418n^5\\
&\qquad+1311365n^4+3790503n^3+460056n^2-271701n-60291)A_n\\
&\quad-4(2n-1)(3874492n^8-2617900n^7-3144314n^6+2947148n^5+647130n^4\\
&\qquad-1182926n^3+115771n^2+170716n-44541)A_{n-1}\\
&\quad-4(n-1)^4(2n-1)(2n-3)(41218n^3+75195n^2+46746n+9898)A_{n-2}=0
\end{aligned}\tag{10.1}$$

for simultaneous approximations to $\zeta(3)$ and $\zeta(5)$ (see [Zu2]), and

$$\begin{aligned}
&n(n+1)^5(91n^3-182n^2+126n-30)A_{n+1}\\
&\quad-n(3458n^8+1729n^7-2947n^6-2295n^5+901n^4+1190n^3\\
&\qquad+52n^2-228n-60)A_n\\
&\quad-(153881n^9-307762n^8+185311n^7+2960n^6-31631n^5-88n^4\\
&\qquad+5239n^3-610n^2-440n+100)A_{n-1}\\
&\quad+24(n-1)^3(2n-1)(6n-5)(6n-7)(91n^3+91n^2+35n+5)A_{n-2}=0
\end{aligned}\tag{10.2}$$

for the sequence $\sum_{k=0}^{n}\binom{n}{k}^6$ (see [Pe], [SJ]).

The differential equations for generating series of (10.1) and (10.2) are of order 9; nevertheless, `Maple` easily factorise the corresponding differential operators and we get the 6th-order linear differential equations

$$\begin{aligned}
&z^5(196z-87)^3(16z^3+752z^2-2368z-1)\frac{\mathrm{d}^6}{\mathrm{d}z^6}\\
&\quad+3z^4(196z-87)^2(21952z^4+873216z^3-2778608z^2+1235312z+435)\frac{\mathrm{d}^5}{\mathrm{d}z^5}\\
&\quad+z^3(196z-87)(85898176z^5+2768881024z^4-8828169756z^3\\
&\qquad+7144975624z^2-1768825884z-491985)\frac{\mathrm{d}^4}{\mathrm{d}z^4}\\
&\quad+2z^2(20932110080z^6+508609400320z^5-1613572776144z^4+1739040695000z^3\\
&\qquad-824264516904z^2+148195933632z+29632635)\frac{\mathrm{d}^3}{\mathrm{d}z^3}\\
&\quad+z(35509291776z^6+579712191744z^5-1530351585392z^4+1500993519824z^3\\
&\qquad-731658297456z^2+173252093886z+20413593)\frac{\mathrm{d}^2}{\mathrm{d}z^2}
\end{aligned}$$

$$
\begin{aligned}
&+ (7138000128z^6 + 55844570880z^5 - 123313425872z^4 + 96429989856z^3 \\
&\qquad - 16021623504z^2 + 17983065996z + 658503)\frac{\mathrm{d}}{\mathrm{d}z} \\
&- 6(15059072z^5 - 4148928z^4 + 9924264z^3 + 214891044z^2 \\
&\qquad + 106071966z + 4609521) = 0
\end{aligned} \tag{10.3}
$$

and

$$
\begin{aligned}
&z^4(z-1)(64z-1)(27z+1)(75z^3+1420z^2+561z+9)\frac{\mathrm{d}^6}{\mathrm{d}z^6} \\
&\quad + z^3(2786400z^6 + 52926750z^5 - 22883417z^4 - 19427551z^3 \\
&\qquad - 654306z^2 + 1308z + 126)\frac{\mathrm{d}^5}{\mathrm{d}z^5} \\
&\quad + 5z^2(3822480z^6 + 77258112z^5 - 18886036z^4 - 23855750z^3 \\
&\qquad - 758207z^2 - 1275z + 99)\frac{\mathrm{d}^4}{\mathrm{d}z^4} \\
&\quad + 5z(9906840z^6 + 214908768z^5 - 10697346z^4 - 53272456z^3 \\
&\qquad - 1605389z^2 - 7047z + 117)\frac{\mathrm{d}^3}{\mathrm{d}z^3} \\
&\quad + 2(22370400z^6 + 528293250z^5 + 82763885z^4 - 96071120z^3 \\
&\qquad - 2836582z^2 - 15165z + 72)\frac{\mathrm{d}^2}{\mathrm{d}z^2} \\
&\quad + 6(1684500z^5 + 44517700z^4 + 17102475z^3 \\
&\qquad - 4687335z^2 - 158967z - 540)\frac{\mathrm{d}}{\mathrm{d}z} \\
&\quad + 12(15750z^4 + 503175z^3 + 327205z^2 - 1845z - 1044) = 0,
\end{aligned} \tag{10.4}
$$

respectively. (The corresponding recursions will be 7-term but we do not use this fact.)

The differential equation (10.3) is MUM, so taking the Frobenius basis $y_0, y_1, y_2, \dots, y_5$ we may compute the inverse of $q(z) = \exp(y_1/y_0)$,

$$
\begin{aligned}
z(q) &= q + 230q^2 - 26827q^3 + 24147708q^4 - 23334210874q^5 + 26542920855790q^6 \\
&\qquad - 33500728089853156q^7 + 45492345805504886104q^8 + O(q^9),
\end{aligned}
$$

and pseudo-coupling

$$
\tilde{K}(q) = \left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2\left(\frac{y_2}{y_0}\right) = 1 + \sum_{l=1}^{\infty}\frac{l^2\tilde{N}_l q^l}{1-q^l} \tag{10.5}
$$

where

$$
\begin{gathered}
\tilde{N}_1 = -320, \quad \tilde{N}_2 = 118264, \quad \tilde{N}_3 = -84117876, \quad \tilde{N}_4 = 80349364184, \\
\tilde{N}_5 = -90632838175404, \quad \tilde{N}_6 = 113783008482427048, \\
\tilde{N}_7 = -153937885949108788148, \quad \tilde{N}_8 = 220092295805975113694144, \quad \dots
\end{gathered}
$$

are expected to be all integers.

The case of the (not MUM) differential equation (10.4) is more delicate: the exponents at the point $z=0$ are $0,0,0,0,0,1$, so that we get two linearly independent analytic solutions at the point. The choice of the first three components of the basis at $z=0$ are as follows:

$$\begin{aligned}
y_0 &= 1+2z+66z^2+1460z^3+54850z^4+2031252z^5+86874564z^6 \\
&\quad + 3848298792z^7+180295263810z^8+O(z^9) \\
y_1 &= y_0\log z+6z+201z^2+5114z^3+\frac{392153}{2}z^4+\frac{37695381}{5}z^5 \\
&\quad + \frac{1636192577}{5}z^6+\frac{515198795302}{35}z^7+\frac{19496105427567}{28}z^8+O(z^9), \\
y_2 &= -y_0\frac{\log^2 z}{2}+y_1\log z+\beta\biggl(z+\frac{45}{4}z^2+\frac{7085}{18}z^3+\frac{1750325}{144}z^4+\frac{293631899}{600}z^5 \\
&\quad + \frac{4039916881}{200}z^6+\frac{4451078944741}{4900}z^7+\frac{663063626523441}{15680}z^8\biggr) \\
&\quad + 108z^2+3690z^3+164379z^4+6851805z^5+\frac{3138305727}{10}z^6 \\
&\quad + \frac{73262810391}{5}z^7+\frac{99760681900977}{140}z^8+O(z^9)
\end{aligned}$$

(for the series y_2 we have an additional 'freedom' $\beta\in\mathbb{Q}$). Then we get the expansion

$$z(q)=q-6q^2-135q^3-380q^4-24960q^5-696366q^6-26153302q^7-901888104q^8+O(q^9)$$

and the coefficients in the Lambert expansion (10.5)

$$\begin{gathered}
\tilde N_1=\beta,\quad \tilde N_2=3(\beta+30),\quad \tilde N_3=63(\beta+20),\quad \tilde N_4=1357\beta+43200,\\
\tilde N_5=5(9139\beta+235152),\quad \tilde N_6=3(479743\beta+14248200),\\
\tilde N_7=126(419883\beta+11883130),\quad \tilde N_8=3(653810477\beta+19370535600),\quad \dots
\end{gathered}$$

are expected to be in $\mathbb{Z}\beta+\mathbb{Z}$.

We have several further examples of 6th-order differential equations with the same collection of the exponents at the point $z=0$ as for equation (10.4). In all such examples we get the similar phenomenon of admitting a free parameter β for the coefficients in the pseudo-coupling (10.5). Here we only mention formulas for $\{A_n\}_{n=0,1,\dots}$ in the examples, where the series $y_0=\sum_{n=1}^\infty A_nz^n$ form analytic solutions in $\mathbb{Z}[[z]]$ to the corresponding differential equations:

$$A_n=\frac{(3n)!}{n!^3}\sum_{k=0}^n\binom{n}{k}^4,\qquad A_n=\frac{(4n)!}{n!^4}\sum_{j,k}\binom{n}{j}\binom{n}{k}^2\binom{k}{j}.$$

Similar phenomena seem to be happened in more general situations. For example, the series

$$y_0(z)=\sum_{n=0}^\infty z^n\frac{(4n)!}{n!^4}\sum_{k=0}^n\binom{n}{k}^4$$

gives an analytic solution to the 8th-order linear differential equation $Dy = 0$, where

$$\begin{aligned} D = \theta^6(\theta-1)^2 - 16z\theta^2(2\theta+1)^2(4\theta+1)(4\theta+3)(3\theta^2+3\theta+1) \\ - 256z^2(2\theta+1)(2\theta+3)(4\theta+1)(4\theta+3)^2(4\theta+5)^2(4\theta+7); \end{aligned}$$

a suitable choice of the solution $y_1(z)$ leads to the inversion $z(q)$ of the series $q(z) = \exp(y_1/y_0)$ satisfying (conjecturally!) $z(q) \in \mathbb{Z}[[q]]$. Finally, one can take a free parameter β for the solution $y_2(z)$ giving the pseudo-coupling (10.5) with the coefficients

$$\begin{gathered} \tilde{N}_1 = \beta, \quad \tilde{N}_2 = 199\beta + 308464, \quad \tilde{N}_3 = 17(18671\beta + 18181888), \\ \tilde{N}_4 = 2(233056091\beta + 290786122832), \ \tilde{N}_5 = 3(311572981529\beta + 381333083404544), \\ \tilde{N}_6 = 20205481092650033\beta + 25853328496828357728, \quad \dots \end{gathered}$$

that are expected to be in $\mathbb{Z}\beta + \mathbb{Z}$. Another interesting example is the series

$$y_0(z) = \sum_{n=0}^{\infty} z^n \sum_{k=0}^{n} \binom{n}{k}^7,$$

which also satisfies a (linearly irreducible) 8th-order differential equation $Dy = 0$ with exponents $0, 0, 0, 0, 0, 0, 1, 1$ about $z = 0$.

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Superseeker of Calabi–Yau differential equations

	N_0	$\lvert N_1\rvert$	$\lvert N_3\rvert$	Hadamard product	# in [AESZ, Table A]
1	1	2	8		184
2	1	2	104		41
3	3	3	28		28
4	3	4	44		23
5	1	4	84		84
6	1	4	284		206
7	1	4	3252	(p)∗(p)	3*
8	1	5	100	(a)∗(a)	100
9	1	6	104	(k)∗(k)	4*
10	1	6	325	(a)∗(f)	160
11	1	9	748	(f)∗(f)	165
12	7	10	295		22
13	7	10	508		235
14	1	10	664	(c)∗(c)	103
15	1	10	870		60
16	6	12	140		130
17	2	12	208		46
18	10	12	236		17
19	3	12	644		16
20	1	12	3204	(A)∗(a)	45
21	1	13	2650	(b)∗(b)	101
22	2	16	208	(a)∗(d)	105
23	10	16	304		21
24	1	16	1232	(A)∗(d)	36
25	1	16	1744	(d)∗(d)	107
26	2	16	2000		42
27	2	16	2106	(a)∗(b)	102
28	1	16	3280		56
29	6	18	490		20
30	9	18	3820		199
31	7	18	5676		234
32	20	20	100		205
33	5	20	1820		18
34	1	20	5924	(a)∗(e)	114, 150
35	1	20	8220		25
36	1	21	15894		15
37	3	24	1552		188
38	3	27	14201		216

	N_0	$\lvert N_1\rvert$	$\lvert N_3\rvert$	Hadamard product	# in [AESZ, Table A]
39	1	27	18089	(B)∗(ι)	70
40	1	28	1036	(b)∗(e)	121
41	6	28	1820		27
42	3	28	3892		119
43	1	32	1440	(l)∗(l), (A)∗(β)	10*, 40
44	1	32	7584		201
45	1	32	26016	(C)∗(d), (A)∗(θ)	3, 30, 31, 72
46	3	32	38880	(A)∗(d)	111
47	3	33	3600	(b)∗(c)	113
48	7	35	2184		28
49	9	36	556		183
50	3	36	3020	(d)∗(f)	163
51	3	36	3284	(A)∗(f)	133
52	6	36	3648		185
53	1	36	8076	(B)∗(e)	110
54	1	36	41421	(B)∗(b)	24
55	1	36	128217204	(B)∗(j)	
56	1	37	15270	(g)∗(g)	144
57	13	39	1621		197
58	6	42	2542	(a)∗(c)	104
59	44	44	308		182
60	1	44	22500	(C)∗(g)	139
61	4	48	1424	(b)∗(d)	106
62	1	48	2864	(B)∗(β), (A)∗(h)	8**, 49, 141
63	3	48	11056	(A)∗(c)	58
64	2	48	11664	(B)∗(d)	48
65	1	48	32368	(d)∗(e), (A)∗(ε)	122
66	1	48	73328	(C)∗(d)	38
67	19	49	1761		186
68	21	51	5095		217
69	20	52	1356		203
70	3	52	52284		117
71	1	52	220220		68
72	1	54	40552		50
73	3	54	64744		223
74	1	55	11655		118
75	5	59	22503		224
76	3	60	1684	(A)∗(g)	137
77	1	60	134292		5, 90, 91, 93, 157
78	1	60	307860	(B)∗(i)	

	N_0	$\|N_1\|$	$\|N_3\|$	Hadamard product	# in [AESZ, Table A]
79	1	63	96866	(f)∗(h)	172
80	1	64	23360		116
81	1	64	246848	(C)∗(c)	69
82	2	66	59386	(B)∗(δ)	151
83	4	68	95246	(c)∗(g)	175
84	12	72	3900	(a)∗(f)	160
85	4	72	20708	(B)∗(f)	134
86	3	76	144196		55
87	1	76	415420	(a)∗(i), (C)∗(δ)	152
88	28	80	2912		212
89	23	80	4655		19
90	1	80	104976		233
91	1	80	174096		83
92	1	80	249872		236
93	11	84	9052		198
94	6	84	20848		29
95	2	84	83412		27^{∞}
96	3	84	113304		124^{∞}
97	1	90	151648		73
98	1	92	585396	(C)∗(b)	51
99	1	96	12064	(C)∗(β)	7*, 43
100	3	96	26208	(c)∗(e), (A)∗(α)	39, 120
101	1	100	126580	(b)∗(i), (C)∗(η)	
102	12	108	968	(c)∗(f)	162
103	12	108	4916	(b)∗(f)	161
104	4	108	10472	(B)∗(g)	138
105	6	108	19598	(f)∗(g)	178
106	3	108	62596	(e)∗(f), (A)∗(ζ)	164
107	2	108	81104	(c)∗(h)	169
108	3	108	206716	(C)∗(f)	135
109	1	108	49457556	(D)∗(g)	140
110	1	112	378800		71
111	19	113	8515		202
112	1	117	713814		4
113	17	126	11700		194
114	1	128	382592		220
115	1	128	5282176	(C)∗(α)	37
116	7	129	41441		193
117	12	132	9736	(a)∗(g)	173
118	4	132	52204		32

	N_0	$\lvert N_1\rvert$	$\lvert N_3\rvert$	Hadamard product	# in [AESZ, Table A]
119	4	132	118772	(a)$*$(h)	167
120	7	138	42984		218
121	14	140	14136		26
122	12	144	7312	(c)$*$(d)	123
123	6	144	30896	(d)$*$(g)	176
124	1	160	9310		19
125	2	160	539680	(e)$*$(e), (c)$*$(i), (C)$*$(θ)	$\widehat{3}$, 6*, 115
126	1	160	1956896		6, 75, 96, 146
127	1	160	5870688		76
128	61	163	4795		124
129	4	180	28320	(b)$*$(h)	168
130	4	180	110940	(B)$*$(h)	142
131	1	180	21847076	(D)$*$(f)	136
132	15	186	20300		226
133	47	189	9277		196
134	12	192	156	(b)$*$(g)	174
135	1	196	2993772		33
136	3	204	18628		228
137	1	208	1218192		237
138	1	208	1863312	(d)$*$(i), (C)$*$(ε)	
139	3	220	267636		215
140	3	228	278988	(e)$*$(g), (A)$*$(γ)	44, 77
141	9	234	103520		214
142	34	236	22848		213
143	56	240	6944		59
144	8	240	117056		74
145	1	240	19105840	(D)$*$(d)	65
146	4	252	387464	(B)$*$(ζ)	
147	1	252	1162036		154
148	3	252	1522388		35
149	2	270	835370	(g)$*$(h)	53, 179
150	7	274	281388		208
151	20	276	116324		222
152	29	285	40626		195
153	1	288	2339616	(e)$*$(h)	$\widehat{5}$, 98, 171
154	1	288	96055968	(D)$*$(e), (A)$*$(κ)	112
155	1	291	7935104		230
156	2	324	1502052		17^∞
157	1	324	10792428		11, 95
158	4	336	595280	(d)$*$(h), (B)$*$(ε)	170

	N_0	$\lvert N_1\rvert$	$\lvert N_3\rvert$	Hadamard product	# in [AESZ, Table A]
159	1	372	71562236	(D)∗(a)	62
160	5	379	1364199		232
161	1	384	164736	(m)∗(m)	13*, 47
162	3	384	1546624	(c)∗(i), (C)∗(α)	37
163	14	420	159040		189
164	4	428	485244		187
165	1	432	78259376	(D)∗(c)	64
166	5	444	1501908		210
167	3	460	894404		231
168	3	468	3687996	(f)∗(i), (C)∗(ζ)	
169	17	478	285760		209
170	1	480	16034720	(D)∗(β)	9*, 9**, 67
171	5	492	872164		221
172	1	492	136094428	(a)∗(j), (D)∗(δ)	153
173	15	498	360988		219
174	1	522	9879192	(h)∗(h)	$\widehat{4}$, 145
175	1	575	63441275		1, 79, 87, 128
176	1	612	51318900	(b)∗(j), (D)∗(η)	
177	1	624	43406256		180
178	13	647	942613		99
179	72	684	398428		200
180	1	684	195638820	(D)∗(b)	63
181	1	736	26911072	(e)∗(i)	$\widehat{6}$, 77, 78.97
182	1	864	147560800	(c)∗(j), (D)∗(α)	66
183	1	900	8364884		227
184	1	928	170809536		10, 54
185	1	1008	607849200	(D)∗(h), (B)∗(κ)	143
186	3	1020	15174100	(g)∗(i), (C)∗(γ)	52
187	3	1056	15001120		229
188	1	1116	349462868	(f)∗(j), (D)∗(ζ)	
189	1	1248	683015008		14, 85, 86, 156
190	1	1344	109320512	(h)∗(i)	$\widehat{11}$, 94
191	7	1434	18676572		109
192	1	1488	517984144	(d)∗(j), (D)∗(ε)	
193	4	2300	253765100		148
194	1	2400	2956977632		211
195	1	2450	623291900		$\widehat{1}$, 131
196	1	2484	1327731388	(g)∗(j), (D)∗(γ)	149
197	1	2592	81451104	(C)∗(j)	
198	1	2628	3966805740		8, 92, 125, 158

	N_0	$\lvert N_1\rvert$	$\lvert N_3\rvert$	Hadamard product	# in [AESZ, Table A]
199	1	3488	1142687008	(i)∗(i)	$\widehat{10}$, 155
200	1	3936	10892932064	(D)∗(i)	
201	1	4192	2124587232		55^{∞}
202	2	4900	1246583800		80, 81
203	1	5472	6444589536	(e)∗(j)	$\widehat{14}$, 88, 89
204	1	7776	66942277344		12
205	1	10080	24400330080	(h)∗(j)	$\widehat{8}$, 82, 126, 127, 129
206	1	14752	711860273440		7, 147
207	1	26400	230398034080	(i)∗(j)	$\widehat{12}$
208	1	41184	5124430612320	(D)∗(j), (D)∗(κ)	61
209	1	57760	3869123234080		$\widehat{7}$
210	1	67104	28583248229280		13, 57, 108
211	1	70944	3707752060576		207
212	1	93984	25265152551072		225
213	1	201888	40177844666400	(j)∗(j)	$\widehat{13}$, 166
214	1	231200	1700894366474400		2, 159
215	1	678816	69080128815414048		9
216	1	791200	4288711075194400		$\widehat{2}$
217	1	2710944	302270555492914464		$\widehat{9}$

Comments. We list $|N_1|$ and $|N_3|$. The reason for not using N_2 is that it is not invariant under the transformation $q \mapsto -q$. There are many differential equations in the table that are just transformations (by Proposition 8)

$$y_0(z) \mapsto \frac{1}{1-pz} y_0\left(\left(\frac{z}{1-pz}\right)^r\right)$$

which transforms the Yukawa coupling $K(q) \mapsto K(q^r)$ These differential equations should not be in the table but we did not know this transformation when we found them. In the Superseeker table we identify $K(q)$ and $K(q^r)$ as the instanton numbers are just thinned out.