ONE PARAMETER MODELS OF HOPF ALGEBRAS ASSOCIATED WITH MULTIPLE ZETA VALUES[1]

W. Zudilin

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1. Introduction. Attempts to find algebraic relations over $\mathbb{Q}$ for the numbers

$$\pi,\ \zeta(3),\ \zeta(5),\ \zeta(7),\ \zeta(9),\ \dots, \tag{1}$$

are still unsuccessful. Conjecturely, the numbers (1) are algebraically independent over $\mathbb{Q}$ and it looks quite natural and interesting to consider for positive integers $s_1, s_2, \dots, s_l$ with $s_1 > 1$ values of the l-fold zeta function

$$\zeta(\boldsymbol{s}) = \zeta(s_1, s_2, \dots, s_l) := \sum_{n_1>n_2>\dots>n_l\geqslant 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_l^{s_l}} \tag{2}$$

since the algebraic structure of the relations between these numbers (in comparison with a conjectured empty structure for (1)) is fairly rich. The numbers (2) are called the *multiple zeta values* (MZVs for brevity), or the *multiple harmonic series*, or the *Euler–Zagier numbers*. To each (2) we assign, as usual, two characteristics: the *weight* (or the *degree*) $|\boldsymbol{s}| := s_1 + s_2 + \dots + s_l$ and the *length* $\ell(\boldsymbol{s}) := l$.

To decribe known relations (i.e., numerical identities) over $\mathbb{Q}$ for the numbers (2), we introduce the standard coding of multi-indices $\boldsymbol{s}$ by words (monomials in non-commutative letters) over the alphabet $X = \{x_0, x_1\}$ by the rule

$$\boldsymbol{s} \mapsto x_{\boldsymbol{s}} = x_0^{s_1-1} x_1 x_0^{s_2-1} x_1 \dots x_0^{s_l-1} x_1.$$

Then

$$\zeta(x_{\boldsymbol{s}}) := \zeta(\boldsymbol{s}) \tag{3}$$

[1] *No use is allowed without references to this preprint.*

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for all *convergent* words (i.e., the words starting with x_0 and ending with x_1); respectively, we define the weight (or the degree) $|x_{\boldsymbol{s}}| := |\boldsymbol{s}|$ as the number of letters and the length $\ell(x_{\boldsymbol{s}}) := \ell(\boldsymbol{s})$ as the number of x_1's.

Let $\mathbb{Q}\langle X\rangle = \mathbb{Q}\langle x_0, x_1\rangle$ be the graded $\mathbb{Q}$-algebra (with x_0 and x_1 both of degree 1) of polynomials in non-commutative indeterminates; the underlying graded rational vector space of $\mathbb{Q}\langle X\rangle$ is denoted $\mathfrak{H}$. Let $\mathfrak{H}^1 = \mathbb{Q}\mathbf{1} \oplus \mathfrak{H}x_1$ and $\mathfrak{H}^0 = \mathbb{Q}\mathbf{1} \oplus x_0\mathfrak{H}x_1$, where $\mathbf{1}$ is a unit (the empty word of weight 0) of the algebra $\mathbb{Q}\langle X\rangle$. Then $\mathfrak{H}^1$ can be regarded as a subalgebra of $\mathbb{Q}\langle X\rangle$, in fact the non-commutative polynomial algebra on generators $y_s = x_0^{s-1}x_1$, while $\mathfrak{H}^0$ can be viewed as the graded $\mathbb{Q}$-vector space spanned by the convergent words. Now, we can think of zeta function as the $\mathbb{Q}$-linear map $\zeta\colon \mathfrak{H}^0 \to \mathbb{R}$ defined by the rules $\zeta(\mathbf{1}) = 1$ and (3).

Consider multiplications $⧢$ on $\mathfrak{H}$ and $*$ on $\mathfrak{H}^1$ by requiring that they distribute over addition, that

$$\mathbf{1} ⧢ w = w ⧢ \mathbf{1} = w, \qquad \mathbf{1} * w = w * \mathbf{1} = w \tag{4}$$

for any word w, and that

$$\begin{aligned} x_j u ⧢ x_k v &= x_j(u ⧢ x_k v) + x_k(x_j u ⧢ v), \\ y_j u * y_k v &= y_j(u * y_k v) + y_k(y_j u * v) + y_{j+k}(u * v) \end{aligned} \tag{5}$$

for any words u, v, letters x_j, x_k or generators y_j, y_k of $\mathfrak{H}^1$, respectively. We mention that inductive arguments show the commutativity and the assosiativity of both multiplications; algebras $(\mathfrak{H}, ⧢)$ and $(\mathfrak{H}^1, *)$ can be regarded as graded Hopf algebras.

Proposition 1. *The map ζ is a homomorphism of $(\mathfrak{H}^0, ⧢)$ into $\mathbb{R}$, i.e.,*

$$\zeta(w_1 ⧢ w_2) = \zeta(w_1)\zeta(w_2) \qquad \text{for all} \quad w_1, w_2 \in \mathfrak{H}^0. \tag{6}$$

Proposition 2. *The map ζ is a homomorphism of $(\mathfrak{H}^0, *)$ into $\mathbb{R}$, i.e.,*

$$\zeta(w_1 * w_2) = \zeta(w_1)\zeta(w_2) \qquad \text{for all} \quad w_1, w_2 \in \mathfrak{H}^0. \tag{7}$$

Although these results are classical (see, e.g., [H2], [HO], [W2]), we give an alternative approach to prove them using a differential-difference origin of multiplications $⧢$ and $*$ in conformal functional models of the shuffle and stuffle algebras, respectively; this way is already known for the proof of relations (6). Our approach can be extended to multiplications generalizing the above ones, called the *quasi-shuffle products* in [H3].

Proposition 3. *The map ζ satisfies the relations*

$$\zeta(x_1 \sqcup\!\sqcup w - x_1 * w) = 0 \qquad \textit{for all} \quad w \in \mathfrak{H}^0 \tag{8}$$

(*in particular*, *the argument of* ζ *in* (8) *belongs to* $\mathfrak{H}^0$).

Proof. For detailed proof we refer to Derivation Theorem [H1, Theorem 5.1] and Theorem 4.3 in [HO].

All known relations over $\mathbb{Q}$ between the multiple zeta values follow from identities (6)–(8). Thus, the following conjecture from [W1] looks quite verisimilar.

Waldschmidt's conjecture. *All relations between MZVs follow from* (6)–(8); *equivalently*,

$$\ker\zeta = \{u \sqcup\!\sqcup v - u * v : u \in \mathfrak{H}^1,\ v \in \mathfrak{H}^0\}.$$

2. Shuffle algebra of polylogarithms. To demonstrate relations (6) for MZVs we introduce a notion of the *polylogarithm*

$$\mathrm{Li}_{\boldsymbol{s}}(z) := \sum_{n_1 > n_2 > \cdots > n_l \geqslant 1} \frac{z^{n_1}}{n_1^{s_1} n_2^{s_2} \cdots n_l^{s_l}}, \qquad |z| < 1,$$

for each set of positive integers $s_1, s_2, \dots, s_l$. Obviously, we obtain

$$\mathrm{Li}_{\boldsymbol{s}}(1) = \zeta(\boldsymbol{s}), \qquad \boldsymbol{s} \in \mathbb{Z}^l, \quad s_1 \geqslant 2,\ s_2 \geqslant 1,\ \dots,\ s_l \geqslant 1. \tag{9}$$

As in Section 1 we define the polylogarithm on words $x_{\boldsymbol{s}}$ setting

$$\mathrm{Li}_{x_{\boldsymbol{s}}}(z) := \mathrm{Li}_{\boldsymbol{s}}(z), \qquad \mathrm{Li}_{\mathbf{1}}(z) := 1, \tag{10}$$

and extend this definition by linearity to the graded algebra $\mathfrak{H}^1$ (not $\mathfrak{H}$ since coding allows only *admissible* words that means 'ending with x_1').

Lemma 1. *Let* $w \neq \mathbf{1}$ *be any admissible word* (*i.e.*, *any monomial in* $\mathfrak{H}^1$), x_j *its first letter* (*hence* $w = x_j u$ *for some admissable word* u *or* $u = \mathbf{1}$). *Then*

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Li}_w(z) = \frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Li}_{x_j u}(z) = \omega_j(z)\,\mathrm{Li}_u(z), \tag{11}$$

where

$$\omega_j(z) = \omega_{x_j}(z) := \begin{cases} \dfrac{1}{z} & \textit{if } x_j = x_0, \\ \dfrac{1}{1-z} & \textit{if } x_j = x_1. \end{cases}$$

Proof. Let $w=x_ju=x_{\boldsymbol s}$ for some multi-index $\boldsymbol s$. Then

$$\frac{\mathrm d}{\mathrm dz}\operatorname{Li}_w(z)=\frac{\mathrm d}{\mathrm dz}\operatorname{Li}_{\boldsymbol s}(z)=\frac{\mathrm d}{\mathrm dz}\sum_{n_1>n_2>\dots>n_l\geqslant1}\frac{z^{n_1}}{n_1^{s_1}n_2^{s_2}\cdots n_l^{s_l}},$$
$$=\sum_{n_1>n_2>\dots>n_l\geqslant1}\frac{z^{n_1-1}}{n_1^{s_1-1}n_2^{s_2}\cdots n_l^{s_l}}.$$

Therefore, if $s_1>1$ (i.e., $x_j=x_0$), we obtain

$$\frac{\mathrm d}{\mathrm dz}\operatorname{Li}_{x_0u}(z)=\frac1z\sum_{n_1>n_2>\dots>n_l\geqslant1}\frac{z^{n_1}}{n_1^{s_1-1}n_2^{s_2}\cdots n_l^{s_l}}$$
$$=\frac1z\operatorname{Li}_{s_1-1,s_2,\dots,s_l}(z)=\frac1z\operatorname{Li}_u(z),$$

while in the case $s_1=1$ (i.e., $x_j=x_1$)

$$\frac{\mathrm d}{\mathrm dz}\operatorname{Li}_{x_1u}(z)=\sum_{n_1>n_2>\dots>n_l\geqslant1}\frac{z^{n_1-1}}{n_2^{s_2}\cdots n_l^{s_l}}=\sum_{n_2>\dots>n_l\geqslant1}\frac{1}{n_2^{s_2}\cdots n_l^{s_l}}\sum_{n_1=n_2+1}^{\infty}z^{n_1-1}$$
$$=\frac{1}{1-z}\sum_{n_2>\dots>n_l\geqslant1}\frac{z^{n_2}}{n_2^{s_2}\cdots n_l^{s_l}}=\frac{1}{1-z}\operatorname{Li}_{s_2,\dots,s_l}(z)=\frac{1}{1-z}\operatorname{Li}_u(z).$$

The proof is complete.

Lemma 1 motivates an extended (to the total algebra $\mathfrak H$) definition of the polylogarithm; namely, we define $\operatorname{Li}_{\mathbf 1}(z)=1$ and

$$\operatorname{Li}_w(z)=\begin{cases}\dfrac{\log^s z}{s!} & \text{if } w=x_0^s \text{ for some } s\geqslant1,\\ \displaystyle\int_0^z\omega_j(z)\operatorname{Li}_u(z)\,\mathrm dz & \text{if } w=x_ju \text{ containts letter } x_1\end{cases}\tag{12}$$

for any word $w\in\mathfrak H$. Then Lemma 1 remains true for this extended version of polylogarithm (the new definition coincides with (10) for admissible words); moreover,

$$\lim_{z\to0+0}\operatorname{Li}_w(z)=0\qquad\text{if the word } w \text{ containts letter } x_1.\tag{13}$$

It is easy to verify that the 'new' polylogarithms are real-valued and continious functions in the real interval $(0,1)$.

Lemma 2. *The map $w\mapsto\operatorname{Li}_w(z)$ is a homomorphism of $(\mathfrak H,\sqcup\!\sqcup)$ into $C((0,1);\mathbb R)$.*

Proof. We must check that

$$\operatorname{Li}_{w_1\sqcup\!\sqcup\, w_2}(z)=\operatorname{Li}_{w_1}(z)\operatorname{Li}_{w_2}(z)\qquad\text{for all}\quad w_1,w_2\in\mathfrak H.\tag{14}$$

It is enough to verify relation (14) for *words* $w_1, w_2 \in \mathfrak{H}$. We prove (14) by induction on $|w_1|+|w_2|$; if $w_1 = \mathbf{1}$ or $w_2 = \mathbf{1}$ relation (14) becomes obvious by (4). Otherwise, $w_1 = x_j u$ and $w_2 = x_k v$, hence by Lemma 1 and the inductive hypothesis we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\big(\operatorname{Li}_{w_1}(z)\operatorname{Li}_{w_2}(z)\big) &= \frac{\mathrm{d}}{\mathrm{d}z}\big(\operatorname{Li}_{x_j u}(z)\operatorname{Li}_{x_k v}(z)\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\operatorname{Li}_{x_j u}(z)\cdot\operatorname{Li}_{x_k v}(z) + \operatorname{Li}_{x_j u}(z)\cdot\frac{\mathrm{d}}{\mathrm{d}z}\operatorname{Li}_{x_k v}(z) \\
&= \omega_j(z)\operatorname{Li}_u(z)\operatorname{Li}_{x_k v}(z) + \omega_k(z)\operatorname{Li}_{x_j u}(z)\operatorname{Li}_v(z) \\
&= \omega_j(z)\operatorname{Li}_{u \sqcup\!\sqcup\, x_k v}(z) + \omega_k(z)\operatorname{Li}_{x_j u \sqcup\!\sqcup\, v}(z) \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\big(\operatorname{Li}_{x_j(u \sqcup\!\sqcup\, x_k v)}(z) + \operatorname{Li}_{x_j(x_j u \sqcup\!\sqcup\, v)}(z)\big) \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\operatorname{Li}_{x_j u \sqcup\!\sqcup\, x_k v}(z) \\
&= \frac{\mathrm{d}}{\mathrm{d}z}\operatorname{Li}_{w_1 \sqcup\!\sqcup\, w_2}(z).
\end{aligned}$$

Therefore,

$$\operatorname{Li}_{w_1}(z)\operatorname{Li}_{w_2}(z) = \operatorname{Li}_{w_1 \sqcup\!\sqcup\, w_2}(z) + C. \tag{15}$$

If at least one among the words w_1 and w_2 contains letter x_1, then tending $z \to 0{+}0$ by (12), (13) we obtain $C = 0$; otherwise, the substitution $z = 1$ gives the same result $C = 0$. Hence equality (15) leads us to the required relation (14).

Proof of Proposition 1. Proposition 1 immediately follows from Lemma 2 by the use of (9).

H. N. Minh and M. Petitot in [MP] (see also [MPH]) calculated the monodromy for the differential equations (11) and proved that the homomorphism $w \mapsto \operatorname{Li}_w(z)$ of the shuffle algebra $(\mathfrak{H}, \sqcup\!\sqcup)$ over $\mathbb{C}$ is bijective, i.e., all $\mathbb{C}$-algebraic relations between polylogarithms come from the shuffle product.

3. Quasi-shuffle products. Both multiplications, the shuffle $\sqcup\!\sqcup$ and the stuffle $*$, can be formalized in a following manner due to M. Hoffman's construction of quasi-shuffle Hopf algebras.

We begin with the graded non-commutative polynomial algebra $\mathfrak{A} = \mathcal{K}\langle A\rangle$ over a subfield $\mathcal{K} \subset \mathbb{C}$, where A is a locally finite set of generators (i.e., the set of generators in each positive degree is finite). As usual, we refer to elements of A as letters, and to monomials in the letters as words. For any word w we write $\ell(w)$ for its length (the number of letters it contains) and $|w|$ for its weight or degree (the sum of the degrees of its factors). The unique word of length 0 is $\mathbf{1}$, the empty word. We define a multiplication $\circ$ by requiring that $\circ$ is distribute over addition, that

$$\mathbf{1}\circ w=w\circ\mathbf{1}=w \tag{16}$$

for any word w, and that

$$a_ju\circ a_kv=a_j(u\circ a_kv)+a_k(a_ju\circ v)+[a_j,a_k](u\circ v) \tag{17}$$

for any words u,v and letters $a_j,a_k\in A$, where a function $[\,\cdot\,,\,\cdot\,]\colon\bar A\times\bar A\to\bar A$ ($\bar A:=A\cup\{\mathbf{0}\}$) satisfies

(S0) $[a,\mathbf{0}]=\mathbf{0}$ for all $a\in\bar A$,
(S1) $[a_j,a_k]=[a_k,a_j]$ for all $a_j,a_k\in\bar A$,
(S2) $[[a_j,a_k],a_l]=[a_j,[a_k,a_l]]$ for all $a_j,a_k,a_l\in\bar A$, and
(S3) either $[a_j,a_k]=\mathbf{0}$ or $|[a_k,a_j]|=|a_j|+|a_k|$ for all $a_j,a_k\in A$.

Then $(\mathfrak{A},\circ)$ is a commutative graded $\mathcal{K}$-algebra (see [H3, Theorem 2.1]).

If $[a_j,a_k]=0$ for all letters $a_j,a_k\in A$, then $(\mathfrak{A},\circ)$ is the shuffle algebra as usually defined; in particular case $A=\{x_0,x_1\}$ we obtain the shuffle algebra $(\mathfrak{A},\circ)=(\mathfrak{H},\sqcup\!\sqcup)$ of MZVs (or polylogarithms). The stuffle algebra $(\mathfrak{H}^1,*)$ can be derived from the general construction by the choice of generators $A=\{y_j\}_{j=1}^\infty$ and brackets

$$[y_j,y_k]=y_{j+k}\qquad\text{for integers } j\geqslant1 \text{ and } k\geqslant1.$$

Consider another multiplication $\bar\circ$ defined by the rules

$$\mathbf{1}\mathbin{\bar\circ}w=w\mathbin{\bar\circ}\mathbf{1}=w,$$
$$ua_j\mathbin{\bar\circ}va_k=(u\mathbin{\bar\circ}va_k)a_j+(ua_j\mathbin{\bar\circ}v)a_k+(u\mathbin{\bar\circ}v)[a_j,a_k]$$

instead of (16), (17), respectively. Then $(\mathfrak{A},\bar\circ)$ is also a commutative graded $\mathcal{K}$-algebra.

Proposition 4. *The algebras* $(\mathfrak{A},\circ)$ *and* $(\mathfrak{A},\bar\circ)$ *coincide.*

Remark. Proposition 4 can be easily verified with the use of the commutativity of the multiplications '$\circ$' and '$\bar\circ$'. But our proof of (18) below remains true even if we omit the commutativity condition (S1).

Proof. It is enough to prove that

$$w_1 \circ w_2 = w_1 \bar{\circ}\, w_2 \tag{18}$$

for any words $w_1, w_2 \in \mathcal{K}\langle A\rangle$. Both sides of (18) are homogeneous monomials from $\mathfrak{A}$ of the same length. We prove (18) by induction on $\ell(w_1) + \ell(w_2)$. If $\ell(w_1) = 0$ or $\ell(w_2) = 0$, then (18) is an obvious identity. If $\ell(w_1) = \ell(w_2) = 1$, hence $w_1 = a_1$ and $w_2 = a_2$ are letters, we obtain

$$a_1 \circ a_2 = a_1a_2 + a_2a_1 + [a_1, a_2] = a_1 \bar{\circ}\, a_2.$$

If $\ell(w_1) > 1$ while $\ell(w_2) = 1$, hence $w_1 = a_1ua_2$ and $w_2 = a_3$, by the inductive hypothesis we obtain

$$\begin{aligned}
a_1ua_2 \circ a_3 &= a_1(ua_2 \circ a_3) + a_3a_1ua_2 + [a_1, a_3]ua_2\\
&= a_1(ua_2 \bar{\circ}\, a_3) + a_3a_1ua_2 + [a_1, a_3]ua_2\\
&= a_1((u \bar{\circ}\, a_3)a_2 + ua_2a_3 + u[a_2, a_3]) + a_3a_1ua_2 + [a_1, a_3]ua_2\\
&= a_1((u \circ a_3)a_2 + ua_2a_3 + u[a_2, a_3]) + a_3a_1ua_2 + [a_1, a_3]ua_2\\
&= (a_1(u \circ a_3) + a_3a_1u + [a_1, a_3]u)a_2 + a_1ua_2a_3 + a_1u[a_2, a_3]\\
&= (a_1u \circ a_3)a_2 + a_1ua_2a_3 + a_1u[a_2, a_3]\\
&= (a_1u \bar{\circ}\, a_3)a_2 + a_1ua_2a_3 + a_1u[a_2, a_3]\\
&= a_1ua_2 \bar{\circ}\, a_3.
\end{aligned}$$

Similarly, if $\ell(w_1) > 1$ and $\ell(w_2) > 1$, hence $w_1 = a_1ua_2$ and $w_2 = a_3va_4$, by the inductive hypothesis we obtain

$$\begin{aligned}
a_1ua_2 \circ a_3va_4 &= a_1(ua_2 \circ a_3va_4) + a_3(a_1ua_2 \circ va_4) + [a_1, a_3](ua_2 \circ va_4)\\
&= a_1(ua_2 \bar{\circ}\, a_3va_4) + a_3(a_1ua_2 \bar{\circ}\, va_4) + [a_1, a_3](ua_2 \bar{\circ}\, va_4)\\
&= a_1((u \bar{\circ}\, a_3va_4)a_2 + (ua_2 \bar{\circ}\, a_3v)a_4 + (u \bar{\circ}\, a_3v)[a_2, a_4])\\
&\qquad + a_3((a_1u \bar{\circ}\, va_4)a_2 + (a_1ua_2 \bar{\circ}\, v)a_4 + (a_1u \bar{\circ}\, v)[a_2, a_4])\\
&\qquad + [a_1, a_3]((u \bar{\circ}\, va_4)a_2 + (ua_2 \bar{\circ}\, v)a_4 + (u \bar{\circ}\, v)[a_2, a_4])\\
&= a_1((u \circ a_3va_4)a_2 + (ua_2 \circ a_3v)a_4 + (u \circ a_3v)[a_2, a_4])\\
&\qquad + a_3((a_1u \circ va_4)a_2 + (a_1ua_2 \circ v)a_4 + (a_1u \circ v)[a_2, a_4])\\
&\qquad + [a_1, a_3]((u \circ va_4)a_2 + (ua_2 \circ v)a_4 + (u \circ v)[a_2, a_4])
\end{aligned}$$

$$
\begin{aligned}
&= (a_1(u \circ a_3va_4) + a_3(a_1u \circ va_4) + [a_1, a_3](u \circ va_4))a_2 \\
&\qquad + (a_1(ua_2 \circ a_3v) + a_3(a_1ua_2 \circ v) + [a_1, a_3](ua_2 \circ v))a_4 \\
&\qquad + (a_1(u \circ a_3v) + a_3(a_1u \circ v) + [a_1, a_3](u \circ v))[a_2, a_4] \\
&= (a_1u \circ a_3va_4)a_2 + (a_1ua_2 \circ a_3v)a_4 + (a_1u \circ a_3v)[a_2, a_4] \\
&= (a_1u \,\bar{\circ}\, a_3va_4)a_2 + (a_1ua_2 \,\bar{\circ}\, a_3v)a_4 + (a_1u \,\bar{\circ}\, a_3v)[a_2, a_4] \\
&= a_1ua_2 \,\bar{\circ}\, a_3va_4
\end{aligned}
$$

The proof is complete.

4. Functional model of stuffle algebra. The functional model of the stuffle algebra cannot be characterized in a way similar to polylogarithmic since the rule (5) does not yield any differential structure. Thus, we require a difference operation instead; namely, we take the simplest one

$$Df(t) = f(t-1) - f(t).$$

It is easy to see that

$$D\big(f_1(t)f_2(t)\big) = Df_1(t) \cdot f_2(t) + f_1(t) \cdot Df_2(t) + Df_1(t) \cdot Df_2(t). \tag{19}$$

The inverse operation can be given by the formula

$$Ig(t) = \sum_{n=1}^{\infty} g(t+n)$$

up to a constant term if we restrict some growth condition for $g(t)$ at infinity, for instance, $g(t) = O(1/t^2)$ as $t \to +\infty$.

In a spirit of the proof of Lemma 2, by (5) and (19) we require functions $\omega_j(t)$ satisfying the relations

$$\omega_j(t)\omega_k(t) = \omega_{j+k}(t) \qquad \text{for integers } j \geqslant 1 \text{ and } k \geqslant 1.$$

The simplest example of such functions can be given by the formulae

$$\omega_j(t) = \frac{1}{t^j}, \qquad j = 1, 2, \dots .$$

This enables us to define the functions

$$\mathrm{Ri}_{\boldsymbol{s}}(t) = \mathrm{Ri}_{s_1,\dots,s_{l-1},s_l}(t) := I\bigg(\frac{1}{t^{s_l}}\,\mathrm{Ri}_{s_1,\dots,s_{l-1}}(t)\bigg)$$

by induction on the length of the multi-index $\boldsymbol{s}$. By definition we obtain

$$D\,\mathrm{Ri}_{uy_j}(t) = \frac{1}{t^j}\,\mathrm{Ri}_u(t). \tag{20}$$

Lemma 3. *There holds the equality*

$$\operatorname{Ri}_{\boldsymbol{s}}(t)=\sum_{n_1>\dots>n_{l-1}>n_l\geqslant 1}\frac{1}{(t+n_1)^{s_1}\cdots(t+n_{l-1})^{s_{l-1}}(t+n_l)^{s_l}}; \tag{21}$$

in particular,

$$\operatorname{Ri}_{\boldsymbol{s}}(0)=\zeta(\boldsymbol{s}), \qquad \boldsymbol{s}\in\mathbb{Z}^l, \quad s_1\geqslant 2,\ s_2\geqslant 1,\ \dots,\ s_l\geqslant 1, \tag{22}$$

$$\lim_{t\to+\infty}\operatorname{Ri}_{\boldsymbol{s}}(t)=0, \qquad \boldsymbol{s}\in\mathbb{Z}^l, \quad s_1\geqslant 2,\ s_2\geqslant 1,\ \dots,\ s_l\geqslant 1. \tag{23}$$

Proof. We have

$$\begin{aligned}
\operatorname{Ri}_{\boldsymbol{s}}(t)&=I\biggl(\frac{1}{t^{s_l}}\operatorname{Ri}_{s_1,\dots,s_{l-1}}(t)\biggr)\\
&=I\biggl(\frac{1}{t^{s_l}}\sum_{n_1>\dots>n_{l-1}\geqslant 1}\frac{1}{(t+n_1)^{s_1}\cdots(t+n_{l-1})^{s_{l-1}}}\biggr)\\
&=\sum_{n=1}^{\infty}\frac{1}{(t+n)^{s_l}}\sum_{n_1>\dots>n_{l-1}\geqslant 1}\frac{1}{(t+n_1+n)^{s_1}\cdots(t+n_{l-1}+n)^{s_{l-1}}}\\
&=\sum_{n_1'>\dots>n_{l-1}'>n\geqslant 1}\frac{1}{(t+n_1')^{s_1}\cdots(t+n_{l-1}')^{s_{l-1}}(t+n)^{s_l}},
\end{aligned}$$

which is the required equality (21).

Further, define the multiplication $\bar{*}$ on $\mathfrak{H}^1$ (hence on the subalgebra $\mathfrak{H}^0$) by the formulae

$$\mathbf{1}\mathbin{\bar{*}}w=w\mathbin{\bar{*}}\mathbf{1}=w, \tag{24}$$

$$uy_j\mathbin{\bar{*}}vy_k=(u\mathbin{\bar{*}}vy_k)y_j+(uy_j\mathbin{\bar{*}}v)y_k+(u\mathbin{\bar{*}}v)y_{j+k}$$

instead of (4), (5).

Lemma 4. *The map $w\mapsto\operatorname{Ri}_w(z)$ is a homomorphism of $(\mathfrak{H}^0,\bar{*})$ into $C([0,+\infty);\mathbb{R})$.*

Proof. We must verify that

$$\operatorname{Ri}_{w_1\mathbin{\bar{*}}w_2}(z)=\operatorname{Ri}_{w_1}(z)\operatorname{Ri}_{w_2}(z) \qquad \text{for all} \quad w_1,w_2\in\mathfrak{H}^0. \tag{25}$$

Without loss of generality we restrict ourselves to *words* $w_1,w_2\in\mathfrak{H}^0$ and prove (25) by induction on $\ell(w_1)+\ell(w_2)$. If $w_1=\mathbf{1}$ or $w_2=\mathbf{1}$, the relation (25) becomes obvious by (24). Otherwise, $w_1 = uy_j$ and $w_2 = vy_k$, hence by (19), (20) and the inductive hypothesis we obtain

$$\begin{aligned}
D\bigl(\mathrm{Ri}_{w_1}(t)\,\mathrm{Ri}_{w_2}(t)\bigr) &= D\bigl(\mathrm{Ri}_{uy_j}(t)\,\mathrm{Ri}_{vy_k}(t)\bigr) \\
&= D\,\mathrm{Ri}_{uy_j}(t)\cdot\mathrm{Ri}_{vy_k}(t) + \mathrm{Ri}_{uy_j}(t)\cdot D\,\mathrm{Ri}_{vy_k}(t) \\
&\qquad + D\,\mathrm{Ri}_{uy_j}(t)\cdot D\,\mathrm{Ri}_{vy_k}(t) \\
&= \frac{1}{t^j}\,\mathrm{Ri}_u(t)\,\mathrm{Ri}_{vy_k}(t) + \frac{1}{t^k}\,\mathrm{Ri}_{uy_j}(t)\,\mathrm{Ri}_v(t) + \frac{1}{t^{j+k}}\,\mathrm{Ri}_u(t)\,\mathrm{Ri}_v(t) \\
&= \frac{1}{t^j}\,\mathrm{Ri}_{u\,\bar{*}\,vy_k}(t) + \frac{1}{t^k}\,\mathrm{Ri}_{uy_j\,\bar{*}\,v}(t) + \frac{1}{t^{j+k}}\,\mathrm{Ri}_{u\,\bar{*}\,v}(t) \\
&= D\bigl(\mathrm{Ri}_{(u\,\bar{*}\,vy_k)y_j}(t) + \mathrm{Ri}_{(uy_j\,\bar{*}\,v)y_k}(t) + \mathrm{Ri}_{(u\,\bar{*}\,v)y_{j+k}}(t)\bigr) \\
&= D\,\mathrm{Ri}_{uy_j\,\bar{*}\,vy_k}(t) \\
&= D\,\mathrm{Ri}_{w_1\,\bar{*}\,w_2}(t)
\end{aligned}$$

Therefore,

$$\mathrm{Ri}_{w_1}(t)\,\mathrm{Ri}_{w_2}(t) = \mathrm{Ri}_{w_1\,\bar{*}\,w_2}(t) + C \tag{26}$$

and tending $t \to +\infty$ by (23) we obtain $C = 0$. Thus equality (26) becomes the required identity (25).

Proof of Proposition 2. Proposition 2 immediately follows from Lemma 4 and Proposition 4 by the use of (22).

We underline that our approach for the proof of Proposition 2 is similar to the approach for the proof of Proposition 1.

References

[H1] M. E. Hoffman, *Multiple harmonic series*, Pacific J. Math. **152**:2 (1992), 275–290.

[H2] M. E. Hoffman, *The algebra of multiple harmonic series*, J. Algebra **194** (1997), 477–495.

[H3] M. E. Hoffman, *Quasi-shuffle products*, J. Algebraic Combin. **11** (2000), 49–68; `http://xxx.lanl.gov/abs/math/9907173`.

[HO] M. E. Hoffman and Y. Ohno, *Relations of multiple zeta values and their algebraic expression*, Preprint, 2000; `http://xxx.lanl.gov/abs/math/9907173`.

[MP] H. N. Minh and M. Petitot, *Lyndon words, polylogarithms and Riemann ζ function*, Formal Power Series and Algebraic Combinatorics (Vienna, 1997), Discrete Math. **217**:1–3 (2000), 273–292.

[MPH] H. N. Minh, M. Petitot, and J. van der Hoeven, *Shuffle algebra and polylogarithms*, Formal Power Series and Algebraic Combinatorics (Toronto, June 1998), Preprint, 1998.

[W1] M. Waldschmidt, *Valeurs zêta multiples: une introduction*, J. Théorie des Nombres de Bordeaux **12** (2000), 581–595.

[W2] M. Waldschmidt, *Multiple polylogarithms*, Lectures at Institute of Mathematical Sciences (Chennai, November 2000), http://www.math.jussieu.fr/~miw/articles/ps/mpl.ps.

Moscow Lomonosov State University
Department of Mechanics and Mathematics
Vorobiovy Gory, Moscow 119899 RUSSIA
URL: http://wain.mi.ras.ru/
E-mail address: wadim@ips.ras.ru