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# Ramanujan-type formulae and irrationality measures of some multiples of $\pi$

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**Abstract.** An explicit construction of simultaneous Padé approximations for generalized hypergeometric series and formulae for the quantities $\pi\sqrt{d}$, $d\in\{1,2,3,10005\}$, in terms of these series are used for estimates of irrationality measures of these multiples of $\pi$. Other possible applications are also discussed.

Bibliography: 14 titles.

## Introduction

An important role in the history of the Archimedes constant $\pi$ is played by formulae allowing one to calculate it with high accuracy (these days one speaks about billions of decimals). One class of these formulae are representations obtained by Ramanujan in 1914 [1], among which we must point out first of all the following two examples:

$$\sum_{\nu=0}^{\infty}\frac{(1/4)_\nu(1/2)_\nu(3/4)_\nu}{\nu!^3}(21460\nu+1123)\cdot\frac{(-1)^\nu}{882^{2\nu+1}}=\frac{4}{\pi}\,, \tag{1}$$

$$\sum_{\nu=0}^{\infty}\frac{(1/4)_\nu(1/2)_\nu(3/4)_\nu}{\nu!^3}(26390\nu+1103)\cdot\frac{1}{99^{4\nu+2}}=\frac{1}{2\pi\sqrt{2}}\,; \tag{2}$$

As usual, $(a)_\nu=\Gamma(a+\nu)/\Gamma(a)=a(a+1)\cdots(a+\nu-1)$ for $\nu\geqslant 1$ and $(a)_0=1$ is the Pochhammer symbol (the shifted factorial); here and throughout, 'empty' products are set equal to 1. These formulae have only recently been rigorously substantiated and until now new formulae of Ramanujan type have arisen in connection with modular parametrization of solutions of differential equations [2] and algorithms for hypergeometric series [3]. We present as examples two further formulae, which we shall use in the present paper:

$$\sum_{\nu=0}^{\infty}\frac{(1/3)_\nu(1/2)_\nu(2/3)_\nu}{\nu!^3}(14151\nu+827)\cdot\frac{(-1)^\nu}{500^{2\nu+1}}=\frac{3\sqrt{3}}{\pi}\,, \tag{3}$$

$$\sum_{\nu=0}^{\infty}\frac{(1/6)_\nu(1/2)_\nu(5/6)_\nu}{\nu!^3}(545140134\nu+13591409)\cdot\frac{(-1)^\nu}{53360^{3\nu+2}}=\frac{3}{2\pi\sqrt{10005}}\,; \tag{4}$$

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formula (3) is proved in [2] (formula (1.19)) and (4) is the celebrated Chudnovskys' formula [4], which enabled them to hold the record in the calculation of $\pi$ in 1989–94.

On the left-hand side of each formula (1)–(4) we see a general hypergeometric series

$$f(z) = {}_mF_{m-1}\left( \begin{matrix} a_1, a_2, \dots, a_m \\ b_2, \dots, b_m \end{matrix} \,\middle|\, z \right) = \sum_{\nu=0}^{\infty} \frac{(a_1)_\nu (a_2)_\nu \cdots (a_m)_\nu}{(b_2)_\nu \cdots (b_m)_\nu} \, \frac{z^\nu}{\nu!}$$
$$= \sum_{\nu=0}^{\infty} z^\nu \frac{a(0)a(1)\cdots a(\nu-1)}{b(0)b(1)\cdots b(\nu-1)},$$

where

$$a(x) = (x+a_1)(x+a_2)\cdots(x+a_m), \qquad b(x) = (x+b_1)(x+b_2)\cdots(x+b_m), \quad b_1 = 1. \tag{5}$$

It is well known that $f(z)$ satisfies a homogeneous linear differential equation of order $m$, so that the system of functions

$$f_i(z) = \left(z\frac{\mathrm{d}}{\mathrm{d}z}\right)^i f(z) = \sum_{\nu=0}^{\infty} \nu^i z^\nu \frac{a(0)a(1)\cdots a(\nu-1)}{b(0)b(1)\cdots b(\nu-1)}, \qquad i = 0, 1, \dots, m-1,$$

produces a solution to some system of $m$ linear differential equations of the first order. In [5] one can find an explicit construction of simultaneous Padé approximations for the system of functions $1, f_0(z), f_1(z), \dots, f_{m-1}(z)$; the author also indicates there (omitting details) arithmetic applications for the values of these functions. The aim of the present paper is to obtain estimates of irrationality measures of the quantities on the right-hand sides of (1)–(4). We state these estimates as upper bounds for the irrationality exponent. Recall that the *irrationality exponent* of a real irrational number $\alpha$ is

$$\mu = \mu(\alpha) := \inf\left\{ c \in \mathbb{R} : \text{the inequality } \left|\alpha - \frac{p}{q}\right| \leqslant |q|^{-c} \text{ has} \right.$$
$$\left. \text{finitely many solutions in } p, q \in \mathbb{Z} \right\}.$$

We also point out the equality $\mu(\alpha) = \mu(\alpha^{-1})$ which immediately follows from this definition.

**Theorem.** *The following estimates hold for the irrationality exponents*:

$$\mu(\pi) \leqslant 57.53011083\ldots, \tag{6}$$
$$\mu(\pi\sqrt{2}\,) \leqslant 13.93477619\ldots, \tag{7}$$
$$\mu(\pi\sqrt{3}\,) \leqslant 44.12528464\ldots, \tag{8}$$
$$\mu(\pi\sqrt{10005}\,) \leqslant 10.02136339\ldots. \tag{9}$$

The program we implement below was in effect announced by the Chudnovskys [4], [6], who presented as applications estimates (without detailed proofs) for irrationality measures of $\pi\sqrt{2}$ and $\pi\sqrt{640320} = 8\pi\sqrt{10005}$, which were worse than (7) and (9). We simplify the analytic and the arithmetic presentations of the construction in [5], which enables us to give a compact proof of the theorem refining the results of [4], [6].

It must be pointed out that there exist other methods for obtaining estimates of the measure of irrationality of $\pi$ and some of its multiples, which produce better results. For instance, our estimates (6) and (8) are worse than the inequalities

$$\mu(\pi) \leqslant 8.01604539\ldots, \qquad \mu(\pi\sqrt{3}\,) \leqslant 4.60157912\ldots$$

obtained by Hata [7]. In the general case, for quantities of the form $\pi\sqrt{d}$, where $d$ is a positive integer, one has the estimate

$$\mu(\pi\sqrt{d}\,) \leqslant 10.88248501\ldots,$$

which follows from the inequality $\mu(\pi^2) \leqslant 5.44124250\ldots$ of Rhin and Viola [8]. Hence (7) does not improve the already known estimate either, and only (9) refines it for $d = 10005$. However, we emphasize that since new formulae of Ramanujan type keep occurring, the methods used in the present paper can find further number-theoretic applications.

## § 1. Simultaneous Padé approximations

The construction below will be parametrized by integers $M$ and $N$ such that

$$M < N \leqslant \left(1 + \frac{1}{m}\right)M - 1. \tag{10}$$

Considering the polynomial

$$\begin{aligned} Q(x) &= \sum_{\mu=0}^{M} (-1)^{\mu} \binom{M}{\mu} \frac{b(N-M)b(N-M+1)\cdots b(N-M+\mu-1)}{a(0)a(1)\cdots a(\mu-1)} x^{\mu} \\ &= {}_{m+1}F_m\left( \begin{matrix} -M,\ N-M+b_1,\ N-M+b_2,\ \ldots,\ N-M+b_m \\ a_1,\qquad a_2,\qquad \ldots,\qquad a_m \end{matrix} \,\middle|\, x \right), \end{aligned} \tag{11}$$

we see that

$$\begin{aligned} Q(z^{-1})f_i(z) &= \sum_{l=0}^{\infty} z^{l-M} \sum_{\substack{\mu=0 \\ \mu \geqslant M-l}}^{M} (-1)^{\mu} \binom{M}{\mu} (\mu + l - M)^i \\ &\qquad \times \frac{b(N-M)\cdots b(N-M+\mu-1)}{a(0)\cdots a(\mu-1)} \, \frac{a(0)a(1)\cdots a(\mu+l-M-1)}{b(0)b(1)\cdots b(\mu+l-M-1)} \\ &= \sum_{l=0}^{\infty} c_{l,i} z^{l-M}, \qquad i = 0, 1, \ldots, m-1. \end{aligned} \tag{12}$$

For convenience we shall separately write the resulting formulae for the coefficients $c_{l,i}$ in (12) on each of the following intervals:

(a) for $0 \leqslant l < M$

$$c_{l,i} = \sum_{\mu=M-l}^{M} (-1)^\mu \binom{M}{\mu} (\mu+l-M)^i \times \frac{b(N-M)b(N-M+1)\cdots b(N-M+\mu-1)}{a(\mu+l-M)\cdots a(\mu-1)\cdot b(0)\cdots b(\mu+l-M-1)}\,; \tag{13}$$

(b) for $M \leqslant l \leqslant N$

$$c_{l,i} = \frac{1}{b(0)b(1)\cdots b(N-M-1)} \sum_{\mu=0}^{M} (-1)^\mu \binom{M}{\mu} (\mu+l-M)^i \times a(\mu)\cdots a(\mu+l-M-1)\cdot b(\mu-M+l)\cdots b(\mu-M+N-1); \tag{14}$$

(c) for $l > N$

$$c_{l,i} = \sum_{\mu=0}^{M} (-1)^\mu \binom{M}{\mu} (\mu+l-M)^i \times \frac{a(\mu)\cdots a(\mu+l-M-1)}{b(0)\cdots b(N-M-1)\cdot b(\mu-M+N)\cdots b(\mu-M+l-1)}\,. \tag{15}$$

The next result has an extremely complicated proof in [5]. We borrowed the idea of an elementary proof of it from [9].

**Lemma 1.** *For all $i = 0, 1, \dots, m-1$ and integers $l$ such that $M \leqslant l \leqslant N$ one has* $c_{l,i} = 0$.

*Proof.* The function

$$(\mu+l-M)^i \cdot a(\mu)\cdots a(\mu+l-M-1)\cdot b(\mu-M+l)\cdots b(\mu-M+N-1)$$

is a $\mu$-polynomial of degree at most

$$i + m(l-M) + m(N-l) \leqslant (m-1) + m(N-M) \leqslant -1 + M$$

(we use condition (10) in the last inequality). On the other hand, for each polynomial $P(x)$ of degree less than $M$ one has

$$\sum_{\mu=0}^{M} (-1)^\mu \binom{M}{\mu} P(\mu) = 0,$$

since each derivative of the polynomial

$$\sum_{\mu=0}^{M} (-1)^\mu \binom{M}{\mu} x^\mu = (1-x)^M$$

of order less than $M$ vanishes at $x = 1$. In view of (14), this leads to the required result.

As one consequence, we see that the system

$$R_i(z) = \sum_{l=N+1}^{\infty} c_{l,i} z^{l-M} = Q(z^{-1}) f_i(z) - P_i(z^{-1}), \qquad i = 0, 1, \ldots, m-1, \tag{16}$$

$$P_i(x) = \sum_{l=0}^{M-1} c_{l,i} x^{M-l}, \qquad i = 0, 1, \ldots, m-1,$$

yields simultaneous approximations (Padé approximations for $N+1 = (1+1/m)M$) to the system of functions $1, f_0(z), \ldots, f_{m-1}(z)$. Since our construction depends on the positive integers $M, N$, it will sometimes be important to indicate explicitly this dependence: we shall write

$$Q(x; M, N) = Q(x),$$
$$P_i(x; M, N) = P_i(x), \quad R_i(z; M, N) = R_i(z), \qquad i = 0, 1, \ldots, m-1.$$

In what follows we require the following simple estimate for the remainders of the approximations (16).

**Lemma 2.** *Let $N = (1 + 1/m)M - 1$, assume that the coefficients $a_j, b_j$ in the expansions* (5) *are positive, and that $b_j \geqslant 1$, $j = 1, \ldots, m$. Then for $|z| < 1/2^m$,*

$$\varlimsup_{M\to\infty} \frac{\log |R_i(z; M, N)|}{M} \leqslant \frac{\log |z|}{m} + (m+2)\log 2 \quad \text{for each } i = 0, 1, \ldots, m-1. \tag{17}$$

*In particular, for $|z| < 1/2^{m(m+2)}$,*

$$\varlimsup_{M\to\infty} \frac{\log |R_i(z; M, N)|}{M} < 0 \qquad \text{for each } i = 0, 1, \ldots, m-1. \tag{18}$$

*Proof.* We find estimates for the coefficients in (15). Assuming that $a_j \leqslant A$ for some integer $A \geqslant 1$, $j = 1, \ldots, m$, for $l > N$ we obtain

$$\begin{aligned} |c_{l,i}| &\leqslant \sum_{\mu=0}^{M} \binom{M}{\mu} l^i \cdot \frac{a(M)a(M+1)\cdots a(l-1)}{b(0)b(1)\cdots b(l-M-1)} \\ &\leqslant 2^M l^{m-1} \cdot \left( \frac{(A+M)(A+M+1)\cdots(A+l-1)}{(l-M)!} \right)^m \\ &= 2^M \binom{A+l-1}{l-M}^m l^{m-1} \leqslant 2^{M+m(A+l-1)} l^{m-1}, \qquad i = 0, 1, \ldots, m-1. \end{aligned}$$

Hence for $|z| < 1/2^m$,

$$\begin{aligned} |R_i(z)| &\leqslant \sum_{l=N+1}^{\infty} |z|^{l-M} |c_{l,i}| \leqslant 2^{M+m(A-1)} |z|^{-M} \sum_{l=N+1}^{\infty} l^{m-1} (2^m |z|)^l \\ &< 2^{M+m(A-1)} |z|^{-M} \cdot \frac{(N+m)^m (2^m|z|)^{N+1}}{(1-2^m|z|)^m}, \qquad i = 0, 1, \ldots, m-1, \end{aligned}$$

which yields (since $N = (1 + 1/m)M - 1$) the limit relation (17).

*Remark.* The $z$ variable of the hypergeometric series on the left-hand sides of (1)–(4) satisfies the inequality $|z| < 1/2^{m(m+2)}$ with $m = 3$.

## § 2. Arithmetic aspects

Our further considerations concern the hypergeometric series in (1)–(4); therefore we set

$$m = 3, \qquad M = 3n, \quad N = 4n - 1,$$

where $n$ is an increasing positive integer. We also set

$$Q_n(x) = Q(x; M, N),$$
$$P_{i,n}(x) = P_i(x; M, N), \quad R_{i,n}(z) = R_i(z; M, N), \qquad i = 0, 1, \dots, m-1.$$

In addition, the polynomial $b(x)$ in (5) will always be equal to $(x+1)^3$, whereas for $a(x)$ we shall have the following options:

(I) $a(x) = (x + 1/4)(x + 1/2)(x + 3/4)$;
(II) $a(x) = (x + 1/3)(x + 1/2)(x + 2/3)$;
(III) $a(x) = (x + 1/6)(x + 1/2)(x + 5/6)$.

For $n = 1, 2, \dots$ let $D_n$ be the least common denominator of the coefficients of $Q_n(x)$. In our discussion of the above-listed cases we shall also show that

$$D_n P_{i,n}(x) \in \mathbb{Z}[x], \qquad i = 0, 1, \dots, m-1, \quad n = 1, 2, \dots.$$

(I) In this case the substitution in (11) yields

$$Q_n(x) = \sum_{\mu=0}^{M} (-1)^\mu 4^{4\mu} x^\mu \frac{M!\,(N-M+\mu)!^3}{(M-\mu)!\,(N-M)!^3(4\mu)!}$$

and by formula (13) for the coefficients of the polynomial $P_{i,n}(x)$ we obtain

$$c_{l,i} = 4^{4(M-l)} \sum_{\mu=M-l}^{M} (-1)^\mu (\mu+l-M)^i \frac{(4(\mu+l-M))!}{(\mu+l-M)!^4} \cdot \frac{M!\,(N-M+\mu)!^3}{(M-\mu)!\,(N-M)!^3(4\mu)!}.$$

The factors

$$(\mu+l-M)^i \frac{(4(\mu+l-M))!}{(\mu+l-M)!^4}$$

participating in the last expression are integers, so that the elements of the required sequence $D_n = D_n^{(\mathrm{I})}$ are for each $n$ the least common denominators of the quantities

$$\frac{M!\,(N-M+\mu)!^3}{(M-\mu)!\,(N-M)!^3(4\mu)!} = \frac{(3n)!\,(n+\mu)!^3}{(3n-\mu)!\,n!^3(4\mu)!} \cdot \left(\frac{n}{n+\mu}\right)^3, \qquad \mu = 0, 1, \dots, 3n.$$

Consider now the integer-valued function

$$\varphi(x, y) = \lfloor 3x - y \rfloor + 3\lfloor x \rfloor + \lfloor 4y \rfloor - \lfloor 3x \rfloor - 3\lfloor x + y \rfloor + 3\lambda(x+y) - 3\lambda(x), \quad (19)$$

where

$$\lambda(x)=\begin{cases}1 & \text{if } \{x\}=0,\\ 0 & \text{otherwise,}\end{cases}$$

and $\lfloor\cdot\rfloor$ and $\{\cdot\}$ are the integer and the fractional parts of a number, respectively. Since for each prime $p$ and each positive integer $N$,

$$\operatorname{ord}_p N! = \left\lfloor\frac{N}{p}\right\rfloor+\left\lfloor\frac{N}{p^2}\right\rfloor+\left\lfloor\frac{N}{p^3}\right\rfloor+\cdots, \quad \operatorname{ord}_p N=\lambda\Bigl(\frac{N}{p}\Bigr)+\lambda\Bigl(\frac{N}{p^2}\Bigr)+\lambda\Bigl(\frac{N}{p^3}\Bigr)+\cdots,$$

we conclude from the definition of $\varphi(x,y)$ that the following equality holds for each $\mu=0,1,\dots,3n$:

$$\prod_p p^{-\varphi(n/p,\mu/p)-\varphi(n/p^2,\mu/p^2)-\cdots}=\frac{(3n)!\,(n+\mu)!^3}{(3n-\mu)!\,n!^3(4\mu)!}\cdot\Bigl(\frac{n}{n+\mu}\Bigr)^3. \tag{20}$$

Hence setting

$$\varphi_0(x)=\max_{0\leqslant y\leqslant 3x}\varphi(x,y), \tag{21}$$

we obtain by (20) the required denominator $D_n$ in the following form:

$$D_n=\prod_p p^{\varphi_0(n/p)+\varphi_0(n/p^2)+\varphi_0(n/p^3)+\cdots}.$$

The function (19) is 1-periodic in each variable, therefore by the prime number theorem the primes $p\leqslant\sqrt{3n}$ make a contribution of order $O(e^{\text{const}\cdot\sqrt{n}})$ to the asymptotic formula for $D_n$ as $n\to\infty$. This means, in particular, that one can consider another sequence

$$\widetilde{D}_n=\prod_{p>\sqrt{3n}}p^{\varphi_0(n/p)},$$

and we have

$$\lim_{n\to\infty}\frac{\log D_n}{n}=\lim_{n\to\infty}\frac{\log\widetilde{D}_n}{n}.$$

Consider now another auxiliary function:

$$\varphi_1(x)=\max_{y\in\mathbb{R}}\varphi(x,y)=\max_{0\leqslant y<1}\varphi(x,y);$$

by contrast with (21) it is 1-periodic and therefore can be explicitly calculated:

$$\varphi_1(x)=\begin{cases}3 & \text{if } \{x\}\in\bigl[\frac14\bigr],\\ 2 & \text{if } \{x\}\in\bigl[0,\frac14\bigr)\cup\bigl(\frac14,\frac13\bigr)\cup\bigl[\frac12\bigr],\\ 1 & \text{if } \{x\}\in\bigl[\frac13,\frac12\bigr)\cup\bigl(\frac12,\frac23\bigr)\cup\bigl[\frac34\bigr],\\ 0 & \text{if } \{x\}\in\bigl[\frac23,\frac34\bigr)\cup\bigl(\frac34,1\bigr)\end{cases}$$

(here $[x]$ is the singleton consisting of $x$). At the same time, for $x \geqslant 1/3$ we obviously have $\varphi_0(x) = \varphi_1(x)$. Hence it remains to calculate the function (21) on the set $0 \leqslant x < 1/3$:

$$\varphi_0(x) = \begin{cases} 3 & \text{if } x \in \left[\frac{1}{4}\right], \\ 2 & \text{if } x \in \left[\frac{1}{6}, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right), \\ 1 & \text{if } x \in \left[\frac{1}{12}, \frac{1}{6}\right), \\ 0 & \text{if } x \in \left[0, \frac{1}{12}\right). \end{cases}$$

As a result, for the evaluation of the asymptotic behaviour of

$$\widetilde{D}_n = \prod_{p>3n} p^{\varphi_0(n/p)} \cdot \prod_{\sqrt{3n}<p\leqslant 3n} p^{\varphi_1(n/p)}$$

as $n \to \infty$ it remains to use the prime number theorem again:

$$\begin{aligned} \lim_{n\to\infty} \frac{\log \widetilde{D}_n}{n} &= \int_0^{1/3} \varphi_0(x)\,\mathrm{d}\biggl(-\frac{1}{x}\biggr) + \int_0^{1/3} \varphi_1(x)\,\mathrm{d}\biggl(\psi(x) + \frac{1}{x}\biggr) \\ &\quad + \int_{1/3}^{1} \varphi_1(x)\,\mathrm{d}\psi(x) \\ &= 18 + 2\gamma + \psi\biggl(\frac{1}{3}\biggr) + \psi\biggl(\frac{2}{3}\biggr) = 18 - 3\log 3 = 14.70416313\dots, \end{aligned}$$

where $\psi(\,\cdot\,)$ is the logarithmic derivative of the gamma function and $\gamma = -\psi(1)$ is Euler's constant.

Summarizing, we can state the following definitive result.

**Lemma 3.** *The asymptotic behaviour of the least common denominator* $D_n = D_n^{(\mathrm{I})}$ *of the polynomials* $Q_n(x)$ *and* $P_{i,n}(x)$, $i = 0, 1, \dots, m$, *is described in case* (I) *by the limit relation*

$$\lim_{n\to\infty} \frac{\log D_n^{(\mathrm{I})}}{n} = 18 - 3\log 3 = 14.70416313\dots.$$

(II) Performing calculations in accordance with formulae (11) and (13) we see that $D_n = D_n^{(\mathrm{II})}$ is the least common denominator of the quantities

$$\frac{M!\,(N-M+\mu)!^3\mu!}{(M-\mu)!\,(N-M)!^3(2\mu)!\,(3\mu)!} = \frac{(3n)!\,(n+\mu)!^3\mu!}{(3n-\mu)!\,n!^3(2\mu)!\,(3\mu)!} \cdot \biggl(\frac{n}{n+\mu}\biggr)^3, \qquad \mu = 0, 1, \dots, 3n.$$

Hence the auxiliary function $\varphi(x, y)$ has the following form:

$$\varphi(x,y) = \lfloor 3x-y\rfloor + 3\lfloor x\rfloor + \lfloor 2y\rfloor + \lfloor 3y\rfloor - \lfloor 3x\rfloor - 3\lfloor x+y\rfloor - \lfloor y\rfloor + 3\lambda(x+y) - 3\lambda(x),$$

and the calculation of the corresponding $\varphi_0(x)$ (for $0 \leqslant x < 1/3$) and $\varphi_1(x)$ produces the following result:

$$\varphi_0(x) = \begin{cases} 3 & \text{for } x \in \left[\frac{2}{9}, \frac{1}{3}\right), \\ 2 & \text{for } x \in \left[\frac{1}{6}, \frac{2}{9}\right), \\ 1 & \text{for } x \in \left[\frac{1}{9}, \frac{1}{6}\right), \\ 0 & \text{for } x \in \left[0, \frac{1}{9}\right), \end{cases}$$

$$\varphi_1(x) = \begin{cases} 3 & \text{for } \{x\} \in \left[\frac{2}{9}, \frac{1}{3}\right), \\ 2 & \text{for } \{x\} \in \left[0, \frac{2}{9}\right) \cup \left[\frac{1}{3}\right] \cup \left[\frac{1}{2}\right], \\ 1 & \text{for } \{x\} \in \left(\frac{1}{3}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{2}{3}\right), \\ 0 & \text{for } \{x\} \in \left[\frac{2}{3}, 1\right). \end{cases}$$

Thus,

$$\begin{aligned} &\int_0^{1/3} \varphi_0(x)\,\mathrm{d}\left(-\frac{1}{x}\right) + \int_0^{1/3} \varphi_1(x)\,\mathrm{d}\left(\psi(x) + \frac{1}{x}\right) + \int_{1/3}^{1} \varphi_1(x)\,\mathrm{d}\psi(x) \\ &\qquad = 15 + 2\gamma - \psi\left(\frac{2}{9}\right) + 2\psi\left(\frac{1}{3}\right) + \psi\left(\frac{2}{3}\right) \\ &\qquad = 15 - \frac{9}{2}\log 3 - \frac{\pi}{2\sqrt{3}} - \gamma - \psi\left(\frac{2}{9}\right) = 13.33336442\ldots, \end{aligned}$$

and we arrive at the following result.

**Lemma 4.** *The asymptotic behaviour of the denominators $D_n = D_n^{(\mathrm{II})}$ is described by the limit relation*

$$\lim_{n\to\infty} \frac{\log D_n^{(\mathrm{II})}}{n} = 15 - \frac{9}{2}\log 3 - \frac{\pi}{2\sqrt{3}} - \gamma - \psi\left(\frac{2}{9}\right) = 13.33336442\ldots.$$

*Remark.* The well-known Gauss formula (see, for example, [10], p. 19) enables one to calculate the value of $\psi(x)$ in terms of elementary functions, however the corresponding expressions are fairly cumbersome; for instance,

$$\begin{aligned} -\psi\left(\frac{2}{9}\right) &= \gamma + \frac{5}{2}\log 3 + \frac{\pi}{2}\,\cot\frac{2\pi}{9} + 2\cos\frac{\pi}{9}\,\log\left(2\sin\frac{2\pi}{9}\right) \\ &\qquad - 2\cos\frac{2\pi}{9}\,\log\left(2\sin\frac{4\pi}{9}\right) - 2\cos\frac{4\pi}{9}\,\log\left(2\sin\frac{\pi}{9}\right). \end{aligned}$$

(III) Arguing as before we conclude that for each $n$ the quantity $D_n = D_n^{(\mathrm{III})}$ is the least common denominator of the quantities

$$\frac{M!\,(N-M+\mu)!^3(3\mu)!}{(M-\mu)!\,(N-M)!^3\mu!\,(6\mu)!} = \frac{(3n)!\,(n+\mu)!^3(3\mu)!}{(3n-\mu)!\,n!^3\mu!\,(6\mu)!}\cdot\left(\frac{n}{n+\mu}\right)^3, \qquad \mu = 0, 1, \ldots, 3n.$$

Hence

$$\varphi(x,y)=\lfloor 3x-y\rfloor+3\lfloor x\rfloor+\lfloor y\rfloor+\lfloor 6y\rfloor-\lfloor 3x\rfloor-3\lfloor x+y\rfloor-\lfloor 3y\rfloor+3\lambda(x+y)-3\lambda(x),$$

therefore

$$\varphi_0(x)=\begin{cases}2 & \text{for } x\in\bigl[\frac16,\frac13\bigr),\\ 1 & \text{for } x\in\bigl[\frac1{18},\frac16\bigr), \qquad 0\leqslant x<\frac13,\\ 0 & \text{for } x\in\bigl[0,\frac1{18}\bigr),\end{cases}$$
$$\varphi_1(x)=\begin{cases}2 & \text{for } \{x\}\in\bigl[0,\frac13\bigr)\cup\bigl[\frac12\bigr],\\ 1 & \text{for } \{x\}\in\bigl[\frac13,\frac12\bigr)\cup\bigl(\frac12,\frac23\bigr)\cup\bigl[\frac{13}{18},\frac56\bigr],\\ 0 & \text{for } \{x\}\in\bigl[\frac23,\frac{13}{18}\bigr)\cup\bigl(\frac56,1\bigr),\end{cases}$$

so that

$$\begin{aligned}\int_0^{1/3}\varphi_0(x)\,\mathrm{d}\biggl(-\frac1x\biggr)+\int_0^{1/3}\varphi_1(x)\,\mathrm{d}\biggl(\psi(x)+\frac1x\biggr)+\int_{1/3}^1\varphi_1(x)\,\mathrm{d}\psi(x)\\ =24+2\gamma+\psi\biggl(\frac13\biggr)+\psi\biggl(\frac23\biggr)+\psi\biggl(\frac56\biggr)-\psi\biggl(\frac{13}{18}\biggr)\\ =24-3\log 3+\psi\biggl(\frac56\biggr)-\psi\biggl(\frac{13}{18}\biggr)=20.97202138\dots,\end{aligned}$$

and we arrive at the following result.

**Lemma 5.** *The asymptotic behaviour of the denominators $D_n=D_n^{(\mathrm{III})}$ is described by the limit relation*

$$\lim_{n\to\infty}\frac{\log D_n^{(\mathrm{III})}}{n}=24-3\log 3+\psi\biggl(\frac56\biggr)-\psi\biggl(\frac{13}{18}\biggr)=20.97202138\dots.$$

## § 3. Asymptotic behaviour of approximations

We point out straight away that in the original paper [5] the author puts forward effective methods for the calculation of the asymptotic behaviour of the polynomials $Q_n(z^{-1})$ and the remainders $R_{i,n}(z)$ as $n\to\infty$. However — and this was precisely the case in [5] — a thorough substantiation of empirical arguments remains outside the reach of a middle-sized paper, therefore we present in what follows another approach to the solution of the above-mentioned problem.

By the general theory of Wilf–Zeilberger all the above-constructed objects — the polynomials $Q_n(x)$, $P_{i,n}(x)$, and the linear forms $R_{i,n}(z)$ — satisfy difference equations with respect to the positive integer parameter $n$. The most convenient (from the standpoint of algorithms at any rate) are the polynomials $Q_n(x)$, which have a simple representation (11) as hypergeometric series. The next result is in effect a special case of Theorem 4.4.1 in [11]. Here we denote by $\mathcal{N}$ the shift operator in the $n$-variable.

**Lemma 6.** *There exists a difference operator*

$$\Delta = F_0(x,n) + F_1(x,n)\mathcal{N} + \cdots + F_s(x,n)\mathcal{N}^s \in \mathbb{Z}[x,n][\mathcal{N}]$$

*such that*

$$\Delta Q_n(x) = F_0(x,n)Q_n(x) + F_1(x,n)Q_{n+1}(x) + \cdots + F_s(x,n)Q_{n+s}(x) = 0, \quad n = 1, 2, \ldots .$$

**Lemma 7.** *Let $\Delta = \Delta(x,n,\mathcal{N})$ be the difference operator of Lemma 6. Then for each positive integer $n \geqslant n_0$ and for $i = 0, 1, \ldots, m-1$,*

$$\Delta(x,n,\mathcal{N})P_{i,n}(x) = 0, \qquad \Delta(z^{-1},n,\mathcal{N})R_{i,n}(z) = 0.$$

*In other words, the sequences of remainders of approximation satisfy the same difference equation as the sequence of denominators $Q_n(x)$ of the approximants.*

*Proof.* For each $n$ we set

$$\begin{aligned}\widetilde{P}_{i,n}(x) &= \Delta(x,n,\mathcal{N})P_{i,n}(x) \\ &= \sum_{j=0}^{s} F_j(x,n)P_{i,n+j}(x) \in \mathbb{Q}[x], \qquad i = 0, 1, \ldots, m-1. \end{aligned} \tag{22}$$

Then

$$\begin{aligned}\widetilde{P}_{i,n}(z^{-1}) &= \Delta(z^{-1},n,\mathcal{N})P_{i,n}(z^{-1}) \\ &= \Delta(z^{-1},n,\mathcal{N})\big(Q_n(z^{-1})f_i(z) - R_{i,n}(z)\big) = -\Delta(z^{-1},n,\mathcal{N})R_{i,n}(z) \\ &= -\sum_{j=0}^{s} F_j(z^{-1},n)R_{i,n+j}(z), \qquad i = 0, 1, \ldots, m-1. \end{aligned} \tag{23}$$

The total degrees of the polynomials $F_j(x,n)$ are bounded by an absolute constant, therefore using in the domain $|z| < 1/2^m = 1/2^3$ the estimates for the $R_{i,n+j}(z)$, $j = 0, 1, \ldots, s$, following from limit relation (17) in Lemma 2 we conclude that for each $\varepsilon > 0$ there exists $n_1 = n_1(\varepsilon)$ such that

$$\frac{\log|\widetilde{P}_{i,n}(z^{-1})|}{n} < \log|z| + 3 \cdot 5\log 2 + \varepsilon \tag{24}$$

for all $n \geqslant n_1(\varepsilon)$. We now select $\varepsilon = \log 2$ and carry out further reasonings for each $n \geqslant n_0 = n_1(\log 2)$ and $i = 0, 1, \ldots, m-1$. Replacing $x = z^{-1}$ by integers $q > 2^3$, by (24) (with $\varepsilon = \log 2$) we obtain

$$|\widetilde{P}_{i,n}(q)| < 2^{16n}q^{-n}. \tag{25}$$

At the same time the polynomial (22) has a fixed denominator after multiplication by which the quantities on the left-hand side of (25) become integers.

Hence $\widetilde{P}_{i,n}(q)=0$ for all $q\geqslant q_0$. We see that the polynomial $\widetilde{P}_{i,n}(x)$ vanishes at infinitely many points, so that $\widetilde{P}_{i,n}(x)\equiv 0$ for each $n\geqslant n_0$ and $i=0,1,\dots,m-1$. In view of the definition (22) of $\widetilde{P}_{i,n}(x)$ and relations (23), we arrive at the required result.

In fact, for the calculation of the difference operator of Lemma 6 there exists the Gosper–Zeilberger algorithm of creative telescoping [11], Chapter 6. Using it, we obtain in case (I) an (explicit, albeit very cumbersome) formula for a difference operator $\Delta=\Delta^{(\mathrm{I})}(x,n,\mathcal{N})$ such that

$$\deg_x\Delta^{(\mathrm{I})}=6,\qquad \deg_n\Delta^{(\mathrm{I})}=33,\qquad \deg_{\mathcal{N}}\Delta^{(\mathrm{I})}=4.$$

The most important information for the evaluation of the asymptotic behaviour is contained in the characteristic polynomial of this difference operator:

$$\begin{aligned}\chi(\mathcal{N})=\chi^{(\mathrm{I})}(\mathcal{N})&=\lim_{n\to\infty}\frac{\Delta^{(\mathrm{I})}(x,n,\mathcal{N})}{n^{33}}\\ &=2^{31}3^3(2048x-243)\bigl(3^9(524288x^2-248832x-59049)(\mathcal{N}^4+1)\\ &\quad+2^2(524288x^2-248832x-59049)\\ &\qquad\times(4194304x^3-5308416x^2+1399680x-19683)\mathcal{N}^3\\ &\quad+2\cdot 3^4(32768x^2+55296x+729)(524288x^2-248832x-59049)\mathcal{N}^2\\ &\quad+2^23^7(64x-9)(524288x^2-248832x-59049)\mathcal{N}\bigr).\end{aligned}\tag{26}$$

In case (II) for the corresponding difference operator $\Delta=\Delta^{(\mathrm{II})}(x,n,\mathcal{N})$ we have

$$\deg_x\Delta^{(\mathrm{II})}=6,\qquad \deg_n\Delta^{(\mathrm{II})}=41,\qquad \deg_{\mathcal{N}}\Delta^{(\mathrm{II})}=4;$$

and its characteristic polynomial differs from $\chi^{(\mathrm{I})}(\mathcal{N})$ by a constant factor:

$$\chi^{(\mathrm{II})}(\mathcal{N})=\lim_{n\to\infty}\frac{\Delta^{(\mathrm{II})}(x,n,\mathcal{N})}{n^{41}}=\frac{3^{15}}{2^{18}}\chi(\mathcal{N})$$

(which, however, comes as no surprise).

Finally, in case (III) for the difference operator $\Delta=\Delta^{(\mathrm{III})}(x,n,\mathcal{N})$ we have

$$\deg_x\Delta^{(\mathrm{III})}=6,\qquad \deg_n\Delta^{(\mathrm{III})}=41,\qquad \deg_{\mathcal{N}}\Delta^{(\mathrm{III})}=4$$

and

$$\chi^{(\mathrm{III})}(\mathcal{N})=\lim_{n\to\infty}\frac{\Delta^{(\mathrm{III})}(x,n,\mathcal{N})}{n^{41}}=\frac{3^{15}}{2^{2}}\chi(\mathcal{N}).$$

Now, the asymptotic behaviour of the above-constructed approximants (to polynomials and linear forms) is for each fixed $z=1/x$ in the domain $0<|z|<1$ completely determined by the zeros of the characteristic polynomial $\chi(\mathcal{N})$. This follows by Lemmas 6, 7, and the following generalization of Poincaré's theorem (see [12], Theorem 1).

**Lemma 8.** *Let $u_n$, $n=n_0,n_0+1,\dots$, be a sequence that is a non-trivial solution of a non-degenerate difference equation with characteristic polynomial $\chi(\mathcal{N})$, $\chi(0)\neq 0$. Then there exists an upper limit $\varlimsup_{n\to\infty}|u_n|^{1/n}$, which is equal to the absolute value $|\mathcal{N}_0|$ of a zero of $\chi(\mathcal{N})$. Moreover, if the other zeros of $\chi(N)$ have absolute values distinct from $|\mathcal{N}_0|$, then one can replace the upper limit by the ordinary one.*

## § 4. Irrationality measures

For estimates of the irrationality measure we require the following result, which was established in [13], Proposition 3.3 (see also [7], Remark 2.1 as regards a refinement of the statement).

**Lemma 9.** *Let $\alpha \in \mathbb{R}$ be an irrational number. Assume that a sequence of linear forms $q_n x - p_n$ with integer coefficients from the field of rationals or an imaginary quadratic field satisfies the relations*

$$\lim_{n\to\infty} \frac{\log|q_n|}{n} = C_1, \qquad \varlimsup_{n\to\infty} \frac{\log|q_n\alpha - p_n|}{n} \leqslant -C_0,$$

*where $C_0$ and $C_1$ are real positive numbers. Then $\mu(\alpha) \leqslant 1 + C_1/C_0$.*

*Proof of the theorem.* We consider in detail the estimate (6). For its derivation in case (I), for each $n = 1, 2, \dots$ we set

$$\begin{aligned} P_n(x) &= 21460 P_{1,n}(x) + 1123 P_{0,n}(x), \\ R_n(z) &= 21460 R_{1,n}(z) + 1123 R_{0,n}(z) \\ &= Q_n(z^{-1}) \sum_{\nu=0}^{\infty} \frac{(1/4)_\nu (1/2)_\nu (3/4)_\nu}{\nu!^3} (21460\nu + 1123) z^\nu - P_n(z^{-1}) \end{aligned}$$

and consider the corresponding numerical forms resulting from the substitution $z = -1/882^2$ and multiplication by the common denominator $D_n = D_n^{(\mathrm{I})}$ of the coefficients of the polynomials $Q_n(x)$ and $P_n(x)$:

$$\begin{aligned} r_n &= D_n R_n(-882^{-2}) = D_n Q_n(-882^2) \cdot \frac{4 \cdot 882}{\pi} - D_n P_n(-882^2) \\ &= q_n \cdot \frac{1}{\pi} - p_n \in \mathbb{Z}\frac{1}{\pi} + \mathbb{Z}, \qquad n = 1, 2, \dots; \end{aligned} \tag{27}$$

it is precisely at this point that we use formula (1). Starting from some $n$ the sequences $Q_n(-882^2)$ and $R_n(-1/882^2)$ satisfy the same difference equation with operator $\Delta^{(\mathrm{I})}(-882^2, n, \mathcal{N})$. Here the first sequence is non-trivial; so also is the second sequence because of (27) and since $1/\pi$ is irrational. By Lemma 8 the asymptotic behaviour of these sequences is determined by certain zeros $\mathcal{N}_1$ and $\mathcal{N}_2$ of the characteristic polynomial (26) with $x = -882^2$:

$$\lim_{n\to\infty} \frac{\log|Q_n(-882^2)|}{n} = \log|\mathcal{N}_1|, \qquad \varlimsup_{n\to\infty} \frac{\log|R_n(-1/882^2)|}{n} \leqslant \log|\mathcal{N}_2|.$$

Multiplying out a numerical coefficient one brings the characteristic polynomial to the following form:

$$\mathcal{N}^4 - 401273814916233455620\mathcal{N}^3 + 163210109239302\mathcal{N}^2 - 22127620\mathcal{N} + 1$$

and for its zeros (of which two are real and two are complex conjugate) we have

$$\log|\mathcal{N}_1| = 47.44117569\dots,$$
$$\log|\mathcal{N}_2| = \log|\mathcal{N}_2'| = -15.80349476\dots, \qquad \log|\mathcal{N}_2''| = -15.83418617\dots.$$

In our choice of a zero majorizing the asymptotic expression for the sequence $R_n(-882^{-2})$ we used the trivial estimate (18) of Lemma 2. For the evaluation of the asymptotic behaviour of the linear forms (27) with integer coefficients it now remains to use Lemma 3. In the notation of Lemma 9 we obtain

$$C_0 = -\log|\mathcal{N}_2| - (18 - 3\log 3) = 1.09933162\ldots,$$
$$C_1 = \log|\mathcal{N}_1| + (18 - 3\log 3) = 62.14533883\ldots,$$

therefore

$$\mu(\pi^{-1}) \leqslant 1 + \frac{C_1}{C_0} = 57.53011083\ldots.$$

In view of the equality $\mu(\pi) = \mu(\pi^{-1})$, we arrive at the required estimate (6).

For the derivation of the estimate (7) we use formula (2). In this case the asymptotic behaviour of the sequence $D_n$ as $n \to \infty$ is also described by Lemma 3, and for $z = 1/99^4$ the corresponding characteristic polynomial (26) is (up to multiplication by a numerical coefficient) equal to

$$\mathcal{N}^4 + 7555286417711367256361763 80\mathcal{N}^3$$
$$+ 248860071425393050 2\mathcal{N}^2 + 2732361980\mathcal{N} + 1.$$

Hence

$$C_0 = 20.62568987\ldots - 14.70416313\ldots = 5.92152673\ldots,$$
$$C_1 = 61.88945992\ldots + 14.70416313\ldots = 76.59362305\ldots$$

and we obtain the estimate

$$\mu\biggl(\frac{1}{\pi\sqrt{2}}\biggr) \leqslant 1 + \frac{C_1}{C_0} = 13.93477619\ldots,$$

which yields in effect (7).

The estimates (8) and (9) correspond to the use of formulae (3) (case (II)) and (4) (case (III)), respectively. For $z = -1/500^2$ the characteristic polynomial is as follows:

$$19683\mathcal{N}^4 - 262145327105399680078732\mathcal{N}^3$$
$$+ 331773760512118098\mathcal{N}^2 - 139968078732\mathcal{N} + 19683,$$

so that in case (II),

$$C_0 = 14.66365222\ldots - 13.33336442\ldots = 1.33028779\ldots,$$
$$C_1 = 44.03567538\ldots + 13.33336442\ldots = 57.36903981\ldots,$$

and by Lemma 9 we obtain the estimate (8). Finally, for $z = -1/53360^3$ the characteristic polynomial is

$$19683\mathcal{N}^4 - 588385936994303964231474272217662479260463923987 32\mathcal{N}^3$$
$$+ 122534920953081108757902878834806098\mathcal{N}^2$$
$$- 85062121695608910732\mathcal{N} + 19683;$$

therefore in case (III),

$$C_0 = 34.90377291\ldots - 20.97202138\ldots = 13.93175152\ldots,$$
$$C_1 = 104.71137186\ldots + 20.97202138\ldots = 125.68339324\ldots,$$

and we arrive at the estimate (9). The proof of the theorem is now complete.

## § 5. Final observations

Of course, we could consider another choice of the parameters $M$ and $N$. However, it turns out that in all cases under consideration our choice of these parameters is optimal. In addition, the evaluation of the asymptotic behaviour of the coefficients and the approximating forms becomes much more complicated in the general case — not only technically, but also as regards substantiations. We also point out that one can immediately obtain difference equations for the sequences of polynomials $P_{i,n}(x)$ and forms $R_{i,n}(z)$ since each of these objects is a double hypergeometric series and a suitable version of the algorithm of creative telescoping can be used. However, the corresponding technical realization of this idea requires an enormous amount of time.

As regards other applications of the methods used in the present paper, the following observation due to Guillera [14] appears important. He points out that formulae of the Ramanujan type are closely connected with rapidly convergent series for the logarithms of algebraic numbers and other interesting constants. More precisely, by considering in [14] series of the following kind:

$$F(k)=\sum_{\nu=0}^{\infty}\frac{(1/4+k)_\nu(1/2+k)_\nu(3/4+k)_\nu}{(1+k)_\nu^3}(21460(\nu+k)+1123)\cdot\frac{(-1)^\nu}{882^{2(\nu+k)+1}}$$

(cf. (1)) he discovers that their values have elementary expressions not only for $k=0$, but also for $k=1/2$. For instance, in the above example

$$F(0)=\frac{4}{\pi},\qquad F\biggl(\frac{1}{2}\biggr)=4\log\biggl(\frac{2\cdot 3^{10}}{7^6}\biggr)$$

(the first equality holds by (1)). A large number of other series in the list in [14] allows one to conjecture the existence of a broad class of formulae for mathematically interesting constants to which one can successfully apply the methods of the present paper. Unfortunately, we know of no more or less systematic theory in this area; the existing examples must perhaps be regarded as lucky incidents.

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