HYPERGEOMETRIC SERIES AND APPROXIMATIONS OF MATHEMATICAL CONSTANTS

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Abstract. I will discuss some identities for generalized hypergeometric series that were discovered quite recently in connection with rational approximations to π, π^2 and π^4. A curious thing is that most of these identities have "automatic" proofs (using creative telescoping), and a problem is to provide "human" proofs by means of classical hypergeometric summation and transformation formulas.

Let me say from the beginning that the subject of my lecture is a brief exposition of two topics in number theory, about rational approximations to numbers of the form $\sqrt{d}/\pi$ and $\sqrt{d}/\pi^2$, and about Apéry-like rational approximations to the number $\zeta(4)=\pi^4/90$. These approximations are deeply related to certain unusual transformations of generalized hypergeometric series. Some of these transformations can be shown by means of creative telescoping due to Gosper–Zeilberger, but this does not provide us a way to deduce a general form for such transformations which is really needed to do some new results in number theory. I address the problems of providing human proofs for the transformations and their generalizations to specialists in special functions well represented at this conference.

1. Guillera's generalization of Ramanujan's series for $1/\pi$

In 1914 S. Ramanujan recorded a list of 17 series for $1/\pi$, from which we indicate the simplest one

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^3}{n!^3}(4n+1)\cdot(-1)^n=\frac{2}{\pi} \tag{1}$$

and also two quite impressive examples

$$\sum_{n=0}^{\infty}\frac{(\frac14)_n(\frac12)_n(\frac34)_n}{n!^3}(21460n+1123)\cdot\frac{(-1)^n}{882^{2n+1}}=\frac{4}{\pi}\,, \tag{2}$$

$$\sum_{n=0}^{\infty}\frac{(\frac14)_n(\frac12)_n(\frac34)_n}{n!^3}(26390n+1103)\cdot\frac{1}{99^{4n+2}}=\frac{1}{2\pi\sqrt{2}} \tag{3}$$

which produce rapidly converging (rational) approximations to π. Here

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}a(a+1)\cdots(a+n-1) & \text{for } n\ge1,\\ 1 & \text{for } n=0,\end{cases}$$

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denotes the Pochhammer symbol (the rising factorial). The Pochhammer products occurring in all formulae of this type may be written in terms of binomial coefficients:

$$\frac{(\frac12)_n^3}{n!^3}=2^{-6n}\binom{2n}{n}^3, \qquad \frac{(\frac13)_n(\frac12)_n(\frac23)_n}{n!^3}=2^{-2n}3^{-3n}\binom{2n}{n}\frac{(3n)!}{n!^3},$$
$$\frac{(\frac14)_n(\frac12)_n(\frac34)_n}{n!^3}=2^{-8n}\frac{(4n)!}{n!^4}, \qquad \frac{(\frac16)_n(\frac12)_n(\frac56)_n}{n!^3}=12^{-3n}\frac{(6n)!}{n!^3(3n)!}.$$

Ramanujan's original list was subsequently extended to several other series. Here are two more celebrated examples:

$$\sum_{n=0}^\infty \frac{(\frac13)_n(\frac12)_n(\frac23)_n}{n!^3}(14151n+827)\cdot\frac{(-1)^n}{500^{2n+1}}=\frac{3\sqrt3}{\pi}, \tag{4}$$

$$\sum_{n=0}^\infty \frac{(\frac16)_n(\frac12)_n(\frac56)_n}{n!^3}(545140134n+13591409)\cdot\frac{(-1)^n}{53360^{3n+2}}=\frac{3}{2\pi\sqrt{10005}}. \tag{5}$$

Formula (4) is proven by H. H. Chan, W.-C. Liaw and V. Tan and (5) is the Chudnovskys' famous formula which enabled them to hold the record for the calculation of π in 1989–94. On the left-hand side of each formula (1)–(5) we have linear combinations of a (generalized) hypergeometric series

$${}_mF_{m-1}\left(\begin{matrix}a_1, & a_2, & \dots, & a_m\\ & b_2, & \dots, & b_m\end{matrix}\,\middle|\, z\right)=\sum_{n=0}^\infty \frac{(a_1)_n(a_2)_n\cdots(a_m)_n}{(b_2)_n\cdots(b_m)_n}\,\frac{z^n}{n!} \tag{6}$$

and its derivative at a point close to the origin. The rapid convergence of the series in (2)–(5) may be used for proving the quantitative irrationality of the numbers $\pi\sqrt{d}$ with $d\in\mathbb{N}$.

Although Ramanujan did not indicate how he arrived at his series, he hinted that these series belong to what is now known as 'the theories of elliptic functions to alternative bases'. The first rigorous mathematical proofs of Ramanujan's series and their generalizations were given by the Borweins and Chudnovskys. These proofs are now significantly simplified and reduced to the theory of classical modular forms in recent works of Heng Huat Chan, Yifan Yang and their collaborators. I do not go in details of that proofs (although they also depend on some hypergeometric transformations) but indicate hints of an automatic proof on the example of the simplest Ramanujan's identity (1) given in 1994 by D. Zeilberger (and his automatic collaborator S. B. Ekhad).

Zeilberger's proof goes the following way. One verifies the (terminating) identity

$$\sum_{n=0}^\infty \frac{(1/2)_n^2(-k)_n}{n!^2(3/2+k)_n}(4n+1)(-1)^n=\frac{\Gamma(3/2+k)}{\Gamma(3/2)\Gamma(1+k)} \tag{7}$$

for all *non-negative* integers k. To do this, divide both sides of (7) by the right-hand side and denote the summand on the left by $F(n,k)$; then take

$$G(n,k)=\frac{(2n+1)^2}{(2n+2k+3)(4n+1)}F(n,k)$$

with the motive that $F(n,k+1)-F(n,k)=G(n,k)-G(n-1,k)$, hence $\sum_n F(n,k)$ is a constant, which is seen to be 1 by plugging in $k=0$. Finally, to deduce (1) one takes $k=-1/2$, which is legitimate in view of Carlson's theorem.

If one wishes to use the latter method of proof for other Ramanujan-type formulae, ingenuity is required in order to put the new parameter k in the right place. This was done only recently by J. Guillera (2002–2007), who used the method to prove some other identities of Ramanujan (in those cases when z has only 2 and 3 in its prime decomposition). If somebody doubts the applicability of the method, then take into account that the purely hypergeometric origin of the method and its independence from the elliptic and modular stuff allowed Guillera to prove new generalizations of Ramanujan-type series, namely,

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^5}{n!^5}(20n^2+8n+1)\frac{(-1)^n}{2^{2n}}=\frac{8}{\pi^2}, \tag{8}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^5}{n!^5}(820n^2+180n+1)\frac{(-1)^n}{2^{10n}}=\frac{128}{\pi^2}, \tag{9}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^3(\frac14)_n(\frac34)_n}{n!^5}(120n^2+34n+3)\frac{1}{2^{4n}}=\frac{32}{\pi^2}, \tag{10}$$

and also to find experimentally four additional formulae

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac14)_n(\frac34)_n(\frac16)_n(\frac56)_n}{n!^5}(1640n^2+278n+15)\frac{(-1)^n}{2^{10n}}\overset{?}{=}\frac{256\sqrt3}{3\pi^2}, \tag{11}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac14)_n(\frac34)_n(\frac13)_n(\frac23)_n}{n!^5}(252n^2+63n+5)\frac{(-1)^n}{48^n}\overset{?}{=}\frac{48}{\pi^2}, \tag{12}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac13)_n(\frac23)_n(\frac16)_n(\frac56)_n}{n!^5}(5418n^2+693n+29)\frac{(-1)^n}{80^{3n}}\overset{?}{=}\frac{128\sqrt5}{\pi^2}, \tag{13}$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac18)_n(\frac38)_n(\frac58)_n(\frac78)_n}{n!^5}(1920n^2+304n+15)\frac{1}{7^{4n}}\overset{?}{=}\frac{56\sqrt7}{\pi^2}. \tag{14}$$

As Guillera notices, the series in (12)–(14) are closely related to the series

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3}(28n+3)\frac{(-1)^n}{48^n}=\frac{16}{\pi\sqrt3},$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac16)_n(\frac56)_n}{n!^3}(5418n+263)\frac{(-1)^n}{80^{3n}}=\frac{640\sqrt{15}}{3\pi},$$

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n(\frac14)_n(\frac34)_n}{n!^3}(40n+3)\frac{1}{7^{4n}}=\frac{49}{3\pi\sqrt3},$$

respectively, proven by elliptic or modular methods. However, there is no obvious way to deduce any of formulae (8)–(14) by modular means; the problem lies in the fact that the (Zariski closure of the) projective monodromy group for the corresponding series $F(z)=\sum_{n=0}^{\infty}u_nz^n$ is always $O_5(\mathbb{R})$, which is essentially 'richer' than $O_3(\mathbb{R})$ for classical Ramanujan's series.

There exists also the higher-dimensional identity

$$\sum_{n=0}^{\infty}\frac{(\frac12)_n^7}{n!^7}(168n^3+76n^2+14n+1)\frac{1}{2^{6n}}\overset{?}{=}\frac{32}{\pi^3},$$

discovered by B. Gourevich in 2002 (using an integer relations algorithm).

Guillera also succeeded in finding another generalizations of the formulae for $1/\pi$ and $1/\pi^2$, although parts of his proofs remain incomplete. These formulae are parametric versions of the previous ones; they look like strange transformations of certain hypergeometric series. Let me indicate some of them as a problem.

Problem 1. For any real $a \ge 0$, prove that

$$\begin{aligned}
f_1(a) &= \sum_{n=0}^{\infty} \frac{(a+\frac12)_n^3}{(a+1)_n^3} \big(6(n+a)+1\big) \cdot \frac{1}{4^n} \\
&\overset{?}{=} \frac{2\cdot 4^{a+1/2}}{\pi\cos^2\pi a} \cdot \left(\frac{\Gamma(a+1)\Gamma(\frac12)}{\Gamma(a+\frac12)}\right)^3 + \frac{(4a)^2}{2a-1} \sum_{n=0}^{\infty} \frac{(\frac12)_n(a+\frac12)_n}{(a+1)_n(\frac32-a)_n}, \\
f_2(a) &= \sum_{n=0}^{\infty} \frac{(a+\frac12)_n^3}{(a+1)_n^3} \big(42(n+a)+5\big) \cdot \frac{1}{2^{6n}} \\
&\overset{?}{=} \frac{2\cdot (2^6)^{a+1/2}}{\pi\cos^2\pi a} \cdot \left(\frac{\Gamma(a+1)\Gamma(\frac12)}{\Gamma(a+\frac12)}\right)^3 + \frac{2^7a^2}{2a-1} \sum_{n=0}^{\infty} \frac{(a+\frac12)_n^2}{(2a+1)_n(\frac32-a)_n},
\end{aligned}$$

and

$$\begin{aligned}
F_1(a) &= \sum_{n=0}^{\infty} \frac{(a+\frac12)_n^5}{(a+1)_n^5} \big(20(n+a)^2+8(n+a)+1\big) \cdot \frac{(-1)^n}{4^n} \\
&\overset{?}{=} \frac{2}{\pi\cos\pi a} \cdot \left(\frac{\Gamma(a+1)\Gamma(\frac12)}{\Gamma(a+\frac12)}\right)^2 f_1(a) + \frac{2^5a^3}{2a-1} \sum_{n=0}^{\infty} \frac{(\frac12)_n^2(a+\frac12)_n}{(a+1)_n^2(\frac32-a)_n}, \\
F_2(a) &= \sum_{n=0}^{\infty} \frac{(a+\frac12)_n^5}{(a+1)_n^5} \big(820(n+a)^2+180(n+a)+13\big) \cdot \frac{(-1)^n}{2^{10n}} \\
&\overset{?}{=} \frac{2^{4a+3}}{\pi\cos\pi a} \cdot \left(\frac{\Gamma(a+1)\Gamma(\frac12)}{\Gamma(a+\frac12)}\right)^2 f_2(a) + \frac{2^{11}a^3}{2a-1} \sum_{n=0}^{\infty} \frac{(a+\frac12)_n^3}{(2a+1)_n^2(\frac32-a)_n}, \\
\widetilde F_2(a) &= \sum_{n=0}^{\infty} \frac{(a+\frac12)_n^3(a+\frac14)_n(a+\frac34)_n}{(a+1)_n^5} \big(120(n+a)^2+34(n+a)+3\big) \cdot \frac{1}{2^{4n}} \\
&\overset{?}{=} \frac{2^{-2a+1}}{\pi\cos 2\pi a} \cdot \frac{\Gamma(a+1)^2\Gamma(\frac14)\Gamma(\frac34)}{\Gamma(a+\frac14)\Gamma(a+\frac34)} f_2(a) + \frac{2^9a^3}{4a-1} \sum_{n=0}^{\infty} \frac{(\frac12)_n^3}{(a+1)_n^2(\frac32-2a)_n}.
\end{aligned}$$

Specializing $a=0$ one gets formulae for $1/\pi$ and $1/\pi^2$, respectively. There is no hope that these identities allow elliptic or modular proofs.

2. Apéry-like rational approximations to $\zeta(4)$

In 1978 R. Apéry showed the irrationality of $\zeta(3)$. His rational approximations to the number in question (known nowadays as *Apéry's constant*) have the form $v_n/u_n \in \mathbb{Q}$ for $n = 0, 1, 2, \dots$, where the denominators $\{u_n\} = \{u_n\}_{n=0,1,\dots}$ and numerators $\{v_n\} = \{v_n\}_{n=0,1,\dots}$ satisfy the same polynomial recurrence

$$(n+1)^3u_{n+1} - (2n+1)(17n^2+17n+5)u_n + n^3u_{n-1} = 0$$

with the initial data

$$u_0 = 1, \quad u_1 = 5, \qquad v_0 = 0, \quad v_1 = 6.$$

Then

$$\lim_{n\to\infty}\frac{v_n}{u_n}=\zeta(3)$$

and, surprisingly, the denominators $\{u_n\}$ are integers:

$$u_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2\in\mathbb{Z},\qquad n=0,1,2,\dots, \tag{15}$$

while the numerators $\{v_n\}$ are 'close' to being integers. Since that discovery there was a big search of a way to extend Apéry's proof to other zeta values but, surprisingly, even the case of $\zeta(4)$ (which, of course, known to be irrational) remains open. In 2002, I proposed a construction of Apéry-like approximations to $\zeta(4)$. The approximations are related to the general Bailey's transformation of the ${}_9F_8$ very-well-poised and balanced hypergeometric series. A special choice of the parameters (when the transformation group acts trivially) gives one an Apéry-like recurrence for $\zeta(4)$ (indicated below), while in order to get an arithmetic result (namely, to measure the quality of rational approximations to $\zeta(4)$) one needs to use the full transformation group of order 51840. But then we are faced to a certain arithmetic conjecture about denominators of the rational approximations which remains unproven. Let me explain hypergeometric details of the construction in the simplest (recurrence) case.

For each $n=0,1,2,\dots$, consider the following two rational functions:

$$R_n(t)=(-1)^n\Big(t+\frac{n}{2}\Big)\frac{\prod_{l=1}^{n}(t-l)^2\cdot\prod_{l=1}^{n}(t+n+l)^2}{\prod_{l=0}^{n}(t+l)^4} \tag{16}$$

and

$$\widetilde{R}_n(t)=\frac{n!\prod_{l=1}^{n}(t-l)}{\prod_{l=0}^{n}(t+l)^2}\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{n}\frac{\prod_{l=0}^{n-1}(t-j+l)}{n!}. \tag{17}$$

Problem 2. Prove that the following equality is valid for any $n\ge 0$:

$$-\frac{1}{3}\sum_{\nu=1}^{\infty}\frac{\mathrm{d}R_n(t)}{\mathrm{d}t}\bigg|_{t=\nu}=\frac{1}{6}\sum_{\nu=1}^{\infty}\frac{\mathrm{d}^2\widetilde{R}_n(t)}{\mathrm{d}t^2}\bigg|_{t=\nu}. \tag{18}$$

The series on the left-hand side is the sequence $u_n\zeta(4)-v_n$, $n=0,1,2,\dots$, of rational approximations to $\zeta(4)$ from my 2002 contribution; both the u_n and v_n satisfy the Apéry-like recursion

$$\begin{aligned}(n+1)^5u_{n+1}&-3(2n+1)(3n^2+3n+1)(15n^2+15n+4)u_n\\&-3n^3(3n-1)(3n+1)u_{n-1}=0\qquad\text{for}\quad n\ge 1,\end{aligned} \tag{19}$$

with the initial data $u_0=1$, $u_1=12$ and $v_0=0$, $v_1=13$.

Concerning the right-hand side of (18), write

$$\widetilde{R}_n(t)=\sum_{k=0}^{n}\bigg(\frac{A_k^{(n)}}{(t+k)^2}+\frac{B_k^{(n)}}{t+k}\bigg)=\sum_{k=0}^{n}\bigg(\frac{A_k}{(t+k)^2}+\frac{B_k}{t+k}\bigg), \tag{20}$$

where

$$A_k=\big(\widetilde{R}_n(t)(t+k)^2\big)\big|_{t=-k},\quad B_k=\frac{\mathrm{d}\big(\widetilde{R}_n(t)(t+k)^2\big)}{\mathrm{d}t}\Big|_{t=-k}, \tag{21}$$

and also

$$\sum_{k=0}^{n} B_k = \sum_{k=0}^{n} \operatorname{Res}_{t=-k} \widetilde{R}_n(t) = -\operatorname{Res}_{t=\infty} \widetilde{R}_n(t) = 0.$$

Then

$$\begin{aligned}
\frac{1}{6}\sum_{\nu=1}^{\infty} \frac{\mathrm{d}^2\widetilde{R}_n(t)}{\mathrm{d}t^2}\bigg|_{t=\nu} &= \frac{1}{6}\sum_{\nu=1}^{\infty}\sum_{k=0}^{n}\biggl(\frac{6A_k}{(\nu+k)^4} + \frac{2B_k}{(\nu+k)^3}\biggr) \\
&= \sum_{k=0}^{n} A_k \sum_{\nu=1}^{\infty} \frac{1}{(\nu+k)^4} + \frac{1}{3}\sum_{k=0}^{n} B_k \sum_{\nu=1}^{\infty} \frac{1}{(\nu+k)^3} \\
&= \sum_{k=0}^{n} A_k \biggl(\zeta(4) - \sum_{l=1}^{k}\frac{1}{l^4}\biggr) + \frac{1}{3}\sum_{k=0}^{n} B_k \biggl(\zeta(3) - \sum_{l=1}^{k}\frac{1}{l^3}\biggr) \\
&= \sum_{k=0}^{n} A_k \cdot \zeta(4) + \frac{1}{3}\sum_{k=0}^{n} B_k \cdot \zeta(3) - \sum_{k=0}^{n}\sum_{l=1}^{k}\biggl(\frac{A_k}{l^4} + \frac{B_k}{3l^3}\biggr).
\end{aligned}$$

In view of (21) we see that

$$\frac{1}{6}\sum_{\nu=1}^{\infty} \frac{\mathrm{d}^2\widetilde{R}_n(t)}{\mathrm{d}t^2}\bigg|_{t=\nu} = \widetilde{u}_n\zeta(4) - \widetilde{v}_n, \tag{22}$$

where

$$\widetilde{u}_n = \sum_{k=0}^{n} A_k^{(n)} = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{n} \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{n} \binom{k+j}{n}, \tag{23}$$

$$\widetilde{v}_n = \sum_{k=0}^{n}\sum_{l=1}^{k}\biggl(\frac{A_k^{(n)}}{l^4} + \frac{B_k^{(n)}}{3l^3}\biggr). \tag{24}$$

The equality $u_n = \widetilde{u}_n$ (cf. (23)) for any $n \ge 0$ was first established by C. Krattenthaler and T. Rivoal. I have verified the equality $v_n = \widetilde{v}_n$ (using the recursion for v_n and the representation (24) for $\widetilde{v}_n$) up to $n = 100$. This has led me to the expectation (18).

Note that for applications in number theory we need the general form of identity (18) which is not known (but the general form of the left-hand side is known).

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