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BRIEF COMMUNICATIONS

Diophantine Problems for q-Zeta Values

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1. INTRODUCTION

As usual, quantities depending on a number q and becoming classical objects as $q \to 1$ (at least formally) are regarded as *q-analogs* or *q-extensions*. A possible way to q-extend the values of the Riemann zeta function reads as follows (here $q \in \mathbb{C}$, $|q| < 1$):

$$\zeta_q(k) = \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n = \sum_{\nu=1}^{\infty} \frac{\nu^{k-1} q^\nu}{1-q^\nu} = \sum_{\nu=1}^{\infty} \frac{q^\nu \rho_k(q^\nu)}{(1-q^\nu)^k}, \qquad k = 1, 2, \dots, \tag{1}$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the sum of powers of the divisors and the polynomials $\rho_k(x) \in \mathbb{Z}[x]$ can be determined recursively by the formulas $\rho_1 = 1$ and $\rho_{k+1} = (1 + (k-1)x)\rho_k + x(1-x)\rho_k'$ for $k = 1, 2, \dots$ (see [1, Part 8, Chap. 1, Sec. 8, Problem 75] for the case $k = 2$). Then the limit relations

$$\lim_{\substack{q \to 1 \\ |q|<1}} (1-q)^k \zeta_q(k) = \rho_k(1) \cdot \zeta(k) = (k-1)! \cdot \zeta(k), \qquad k = 2, 3, \dots, \tag{2}$$

hold; the equality $\rho_k(1) = (k-1)!$ is proved in [2, formula (7)]. The above defined q-zeta values (1) present several new interesting problems in the theory of diophantine approximations and transcendental numbers; these problems are extensions of the corresponding problems for ordinary zeta values and we state some of them in Sec. 3 of this note. Our nearest aim is to demonstrate how some recent contributions to the arithmetic study of the numbers $\zeta(k)$, $k = 2, 3, \dots$, successfully work for q-zeta values. Namely, we mean the hypergeometric construction of linear forms (proposed in the works of E. M. Nikishin [3], L. A. Gutnik [4], Yu. V. Nesterenko [5]) and the arithmetic method (due to G. V. Chudnovsky [6], E. A. Rukhadze [7], M. Hata [8]) accompanied by the group-structure scheme (due to G. Rhin and C. Viola [9], [10]). The next section contains new irrationality measures of the numbers $\zeta_q(1)$ and $\zeta_q(2)$ for $q^{-1} = p \in \mathbb{Z} \setminus \{0, \pm 1\}$, and our starting point is the following table illustrating a connection of some objects and their q-extensions (here $\lfloor \cdot \rfloor$ denotes the integral part of a number and the notation 'l.c.m.' means the least common multiple). We refer the reader to the book [11] and the papers [12]–[14], where some motivations and justifications are presented.

ordinary objects	q-extensions, $p = 1/q \in \mathbb{Z} \setminus \{0, \pm 1\}$
numbers $n \in \mathbb{Z}$	'numbers' $[n]_p = \dfrac{p^n - 1}{p - 1} \in \mathbb{Z}[p]$
primes $l \in \{2, 3, 5, 7, \dots\} \in \mathbb{Z}$	irreducible reciprocal polynomials $\Phi_l(p) = \prod_{\substack{k=1 \\ (k,l)=1}}^{l} (p - e^{2\pi i k/l}) \in \mathbb{Z}[p]$

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ordinary objects	q-extensions, $p=1/q\in\mathbb{Z}\setminus\{0,\pm1\}$
Euler's gamma function $\Gamma(t)$	Jackson's q-gamma function $\Gamma_q(t)=\dfrac{\prod_{\nu=1}^{\infty}(1-q^\nu)}{\prod_{\nu=1}^{\infty}(1-q^{t+\nu-1})}\,(1-q)^{1-t}$
the factorial $n!=\Gamma(n+1)$ $n!=\prod_{\nu=1}^{n}\nu\in\mathbb{Z}$	q-factorial $[n]_q!=\Gamma_q(n+1)$ $[n]_p!=\prod_{\nu=1}^{n}\dfrac{p^\nu-1}{p-1}=p^{n(n-1)/2}[n]_q!\in\mathbb{Z}[p]$
$\operatorname{ord}_l n!=\left\lfloor\dfrac{n}{l}\right\rfloor+\left\lfloor\dfrac{n}{l^2}\right\rfloor+\cdots$	$\operatorname{ord}_{\Phi_l(p)}[n]_p!=\left\lfloor\dfrac{n}{l}\right\rfloor,\quad l=2,3,4,\dots$
$D_n=\text{l.c.m.}(1,\dots,n)=\prod_{\text{primes } l\le n} l^{\lfloor\log n/\log l\rfloor}\in\mathbb{Z}$	$D_n(p)=\text{l.c.m.}([1]_p,\dots,[n]_p)=\prod_{l=1}^{n}\Phi_l(p)\in\mathbb{Z}[p]$
the prime number theorem $\lim_{n\to\infty}\dfrac{\log D_n}{n}=1$	Mertens' formula $\lim_{n\to\infty}\dfrac{\log\lvert D_n(p)\rvert}{n^2\log\lvert p\rvert}=\dfrac{3}{\pi^2}$

If $\psi(x)$ is the logarithmic derivative of Euler's gamma function and $\{x\}=x-\lfloor x\rfloor$ is the fractional part of a number x, then, for each semi-interval $[u,v)\subset(0,1)$, Mertens' formula yields the limit relation

$$\lim_{n\to\infty}\frac{1}{n^2\log|p|}\sum_{l\,:\,\{n/l\}\in[u,v)}\log|\Phi_l(p)|=\frac{3}{\pi^2}\bigl(\psi'(u)-\psi'(v)\bigr)=\frac{3}{\pi^2}\int_u^v \mathrm{d}\bigl(-\psi'(x)\bigr) \tag{3}$$

(see [14, Lemma 1]), which can be regarded as a q-extension of the formula

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\substack{\text{primes } l>\sqrt{Cn}\\ \{n/l\}\in[u,v)}}\log l=\psi(v)-\psi(u)=\int_u^v \mathrm{d}\psi(x)$$

in the arithmetic method of [6–10].

2. RATIONAL APPROXIMATIONS TO q-ZETA VALUES AND BASIC TRANSFORMATIONS

Let a_0, a_1, a_2, and b be positive integers satisfying the condition $a_1+a_2\le b$. Then, Heine's series

$$F(\boldsymbol{a},b)=\frac{\Gamma_q(b-a_2)}{(1-q)\Gamma_q(a_1)}\sum_{t=0}^{\infty}\frac{\Gamma_q(t+a_1)\,\Gamma_q(t+a_2)}{\Gamma_q(t+1)\,\Gamma_q(t+b)}\,q^{a_0t}$$

becomes a $\mathbb{Q}(p)$-linear form $F(\boldsymbol{a},b)=A\zeta_q(1)-B$ with the property

$$p^{-M}D_m(p)\cdot F(\boldsymbol{a},b)\in\mathbb{Z}[p]\zeta_q(1)+\mathbb{Z}[p], \tag{4}$$

where $M=M(\boldsymbol{a},b)$ is some (explicitly defined) integer and m is the maximum of the 6-element set

$$c_{00}=a_0+a_1+a_2-b-1,\qquad c_{01}=a_0-1,\qquad c_{11}=a_1-1,\qquad c_{21}=a_2-1,$$
$$c_{12}=b-a_1-1,\qquad c_{22}=b-a_2-1.$$

Taking $H(\boldsymbol{c})=F(\boldsymbol{a},b)$ and using the stability of the quantity

$$\frac{F(a_0,a_1,a_2,b)}{\Gamma_q(a_0)\,\Gamma_q(a_2)\,\Gamma_q(b-a_2)}=\frac{H(\boldsymbol{c})}{\Pi_q(\boldsymbol{c})},\qquad\text{where}\quad \Pi_q(\boldsymbol{c})=[c_{01}]_q!\,[c_{21}]_q!\,[c_{22}]_q!=p^{-N(\boldsymbol{c})}\Pi_p(\boldsymbol{c}),$$

under the action of the transformations

$$\begin{aligned}\tau &= (c_{22}\ c_{21}\ c_{01}\ c_{11}\ c_{12}\ c_{00})\colon (a_0, a_1, a_2, b) \mapsto (a_1, b-a_1, a_0, a_0+a_2),\\ \sigma &= (c_{11}\ c_{21})(c_{12}\ c_{22})\colon \qquad (a_0, a_1, a_2, b) \mapsto (a_0, a_2, a_1, b),\end{aligned}$$

we arrive at the following inclusions (which improve (4)):

$$p^{-M} D_m(p)\Omega^{-1}(p) \cdot F(\boldsymbol{a}, b) \in \mathbb{Z}[p]\zeta_q(1) + \mathbb{Z}[p] \tag{5}$$

with

$$\Omega(p) = \prod_{l=1}^{m} \Phi_l^{\nu_l}(p), \qquad \nu_l = \max_{\mathfrak{g}\in\langle\tau^2,\sigma\rangle} \operatorname{ord}_{\Phi_l(p)} \frac{\Pi_p(\boldsymbol{c})}{\Pi_p(\mathfrak{g}\boldsymbol{c})}. \tag{6}$$

In addition, trivial estimates for $F(\boldsymbol{a}, b)$ and explicit formulas for the coefficient A imply that

$$|F(\boldsymbol{a}, b)| = |p|^{O(b)}, \qquad |A| \le |p|^{(a_0+a_1+a_2)b-(a_1^2+a_2^2+b^2)/2+O(b)} \tag{7}$$

with some absolute constant in $O(b)$.

Note that the nontrivial transformation τ of the quantity $H(\boldsymbol{c})/\Pi_q(\boldsymbol{c})$ has been obtained (in another notation) by E. Heine [15] already in 1847. The transformation group $\mathfrak{G} = \langle\tau, \sigma\rangle$ of order 12 has no ordinary analog, since the corresponding (in the limit as $q \to 1$) Gauss hypergeometric series are divergent. We use the group $\langle\tau^2, \sigma\rangle$ of order 6 instead of the total available group $\mathfrak{G}$ to ensure the required condition $a_1 + a_2 \le b$. Now, choosing $a_0 = a_2 = 8n+1$, $a_1 = 6n+1$, and $b = 15n+2$ and having in mind (5), (7), and (3), we derive the following result.

Theorem 1. *For each $q = 1/p$, $p \in \mathbb{Z}\backslash\{0, \pm 1\}$, the number $\zeta_q(1)$ is irrational and its irrationality exponent satisfies the estimate*

$$\mu(\zeta_q(1)) \le 2.42343562\dots . \tag{8}$$

A value $\mu = \mu(\alpha)$ is said to be the *irrationality exponent* of a real irrational number α if μ is the least possible exponent such that for any $\varepsilon > 0$ the inequality $|\alpha - a/b| \le b^{-(\mu+\varepsilon)}$ has only finitely many solutions in integers a and b. The estimate (8) can be compared with the previous result $\mu(\zeta_q(1)) \le 2\pi^2/(\pi^2-2) = 2.50828476\dots$, of P. Bundschuh and K. Väänänen in [12] corresponding to the choice $a_0 = a_1 = a_2 = n+1$ and $b = 2n+2$ in the above notation.

Similar arguments with a simpler group $\langle\sigma\rangle$ of order 2 can be put forward to improve W. Van Assche's estimate $\mu(\log_q(2)) \le 3.36295386\dots$ in [13] for the following q-extension of $\log(2)$:

$$\log_q(2) = \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}q^\nu}{1-q^\nu} = \sum_{\nu=1}^{\infty} \frac{q^\nu}{1+q^\nu}.$$

Namely, in [14] we obtain the inequality $\mu(\log_q(2)) \le 3.29727451\dots$ for $q^{-1} = p \in \mathbb{Z} \setminus \{0, \pm 1\}$.

In the case of the numbers $\zeta_q(2)$, consider the positive integers $(\boldsymbol{a}, \boldsymbol{b}) = (a_1, a_2, a_3, b_2, b_3)$ satisfying the conditions $a_j < b_k$, $a_1 + a_2 + a_3 < b_2 + b_3$ and the q-basic hypergeometric series

$$\begin{aligned}\widetilde{F}(\boldsymbol{a}, \boldsymbol{b}) &= \frac{\Gamma_q(b_2-a_2)\,\Gamma_q(b_3-a_3)}{(1-q)^2\Gamma_q(a_1)} \sum_{t=0}^{\infty} \frac{\Gamma_q(t+a_1)\,\Gamma_q(t+a_2)\,\Gamma_q(t+a_3)}{\Gamma_q(t+1)\,\Gamma_q(t+b_2)\,\Gamma_q(t+b_3)}\, q^{(b_2+b_3-a_1-a_2-a_3)t}\\ &= \widetilde{A}\zeta_q(2) - \widetilde{B}.\end{aligned}$$

Then $p^{-M}D_{m_1}(p)D_{m_2}(p) \cdot \widetilde{F}(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{Z}[p]\zeta_q(2) + \mathbb{Z}[p]$, where $m_1 \ge m_2$ are the two successive maxima of the 10-element set

$$c_{00} = (b_2+b_3) - (a_1+a_2+a_3) - 1, \qquad c_{jk} = \begin{cases} a_j - 1 & \text{for } k = 1, \\ b_k - a_j - 1 & \text{for } k = 2, 3, \end{cases} \qquad j = 1, 2, 3,$$

and, in addition,

$$|\widetilde{F}(\boldsymbol{a},\boldsymbol{b})| = |p|^{O(\max\{b_2,b_3\})}, \qquad |\widetilde{A}| \le |p|^{b_2b_3-(a_1^2+a_2^2+a_3^2)/2+O(\max\{b_2,b_3\})}.$$

The $\boldsymbol{c}$-permutation group $\mathfrak{G} \subset \mathfrak{S}_{10}$ generated by all permutations of a_1, a_2, a_3, the permutation of b_2, b_3, and the permutation $(c_{00}\ c_{22})(c_{11}\ c_{33})(c_{13}\ c_{31})$, has order 120 and is known in connection with the Rhin–Viola proof [9] of the new irrationality measure for $\zeta(2)$ (see also [16, Sec. 6]). In notation $\widetilde{H}(\boldsymbol{c}) = \widetilde{F}(\boldsymbol{a},\boldsymbol{b})$, the quantity

$$\frac{\widetilde{H}(\boldsymbol{c})}{[c_{00}]_q!\,[c_{21}]_q!\,[c_{22}]_q!\,[c_{33}]_q!\,[c_{31}]_q!}$$

is stable under the action of the group $\mathfrak{G}$. This $\mathfrak{G}$-stability yields the inclusions

$$p^{-M}D_{m_1}(p)D_{m_2}(p)\widetilde{\Omega}^{-1}(p)\cdot\widetilde{F}(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{Z}[p]\zeta_q(2)+\mathbb{Z}[p]$$

with a quantity $\widetilde{\Omega}(p)$ defined as in (6). Finally, choosing $a_1 = 5n+1$, $a_2 = 6n+1$, $a_3 = 7n+1$, and $b_2 = 14n+2$, $b_3 = 15n+2$, we deduce the following result [17].

Theorem 2. *For each $q = 1/p$, $p \in \mathbb{Z}\backslash\{0,\pm1\}$, the number $\zeta_q(2)$ is irrational and its irrationality exponent satisfies the estimate*

$$\mu(\zeta_q(2)) \le 4.07869374\dots\,. \tag{9}$$

Quantitative estimates of type (9) for $\zeta_q(2)$ have not been previously stated, although the transcendence of $\zeta_q(2)$ for any algebraic number q with $0<|q|<1$ follows from Nesterenko's theorem [18].

It is also pleasant to mention that the simpler choice of the parameters $a_1 = a_2 = a_3 = n+1$, $b_2 = b_3 = 2n+2$ also proves the irrationality of $\zeta_q(2)$ for $q^{-1} \in \mathbb{Z}\setminus\{0,\pm1\}$, and the limit $q \to 1$ produces Apéry's original sequence [19] of rational approximations to $\zeta(2)$.

We would like to stress that using, as in [7–10], (multiple) q-integrals for both series $F(\boldsymbol{a}, b)$ and $\widetilde{F}(\boldsymbol{a},\boldsymbol{b})$ in the study of arithmetical properties of the numbers $\zeta_q(1)$ and $\zeta_q(2)$ leads to great difficulties. The reason for this is that no change of variable concept in q-integration (see [20; 21, Sec. 2.2.4]).

3. GENERAL PROBLEMS FOR q-ZETA VALUES

We begin by mentioning that, for an even integer $k \ge 2$, the series $E_k(q) = 1 - 2k\zeta_q(k)/B_k$, where $B_k \in \mathbb{Q}$ are the Bernoulli numbers, is known as the *Eisenstein series*. Therefore, the modular origin (with respect to the parameter $\tau = \log q/2\pi i$) of the functions $E_4, E_6, E_8, \dots$ yields the algebraic independence of the functions $\zeta_q(2), \zeta_q(4), \zeta_q(6)$ over $\mathbb{Q}[q]$, while all other even q-zeta values are polynomials in $\zeta_q(4)$ and $\zeta_q(6)$. In this sense, the consequence of Nesterenko's theorem [18] "the numbers $\zeta_q(2), \zeta_q(4), \zeta_q(6)$ are algebraically independent over $\mathbb{Q}$ for algebraic q, $0<|q|<1$" reads as a complete q-extension of the consequence of Lindemann's theorem [22] "$\zeta(2) = \pi^2/6$ is transcendental." Moreover, the transcendence of values of the function

$$1+4\sum_{\nu=0}^{\infty}\frac{(-1)^\nu q^{2\nu+1}}{1-q^{2\nu+1}} = \biggl(1+2\sum_{n=1}^{\infty}q^{n^2}\biggr)^2 \tag{10}$$

at algebraic points q, $0<|q|<1$, also follows from Nesterenko's theorem (a proof of Jacobi's identity (10) can be found, e.g., in [23, Theorem 2]); the series on the left-hand-side of (10) is the q-analog of the series

$$4\sum_{\nu=0}^{\infty}\frac{(-1)^\nu}{2\nu+1} = \pi.$$

The best known estimate for the irrationality exponent of (10) in the case $q^{-1}\in\mathbb{Z}\setminus\{0,\pm1\}$ was obtained in [24].

The limit relations (2) as well as the expected algebraic structure of the ordinary zeta values motivate the following questions (we also regard $\zeta_q(1)$ to be an odd q-zeta value, although the corresponding ordinary harmonic series is divergent as $q\to1$).

Problem 1. *Prove that the q-zeta values* $\zeta_q(1),\zeta_q(2),\zeta_q(3),\dots$ *as functions of* q *are linearly independent over* $\mathbb{C}(q)$.

Problem 2. *Prove that the q-functional set involving the three even q-zeta values* $\zeta_q(2)$, $\zeta_q(4)$, $\zeta_q(6)$ *and all odd q-zeta values* $\zeta_q(1),\zeta_q(3),\zeta_q(5),\dots$, *consists of functions that are algebraically independent over* $\mathbb{C}(q)$

The associated diophantine problems consist in proving the corresponding linear and algebraic independences over the algebraic closure of $\mathbb{Q}$ for algebraic q with $0<|q|<1$. In this direction, even irrationality and $\mathbb{Q}$-linear independence results for q-zeta values at the point $q\in\mathbb{Q}$ with $q^{-1}\in\mathbb{Z}\setminus\{0,\pm1\}$ would be very interesting.

A problem of another type is to construct a model of multiple q-zeta values involving q-zeta values (1) and possessing properties similar to the model of multiple zeta values [25].

REFERENCES

1. G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Pt. 2, 3rd ed., Springer-Verlag, New York, 1976.
2. M. Kaneko, N. Kurokawa, and M. Wakayama, E-print `math.QA/0206171` (June 2002).
3. E. M. Nikishin, *Mat. Sb.* [*Math. USSR-Sb.*], **109 (151)** (1979), no. 3 (7), 410–417.
4. L. A. Gutnik, *Acta Arith.*, **42** (1983), no. 3, 255–264.
5. Yu. V. Nesterenko, *Mat. Zametki* [*Math. Notes*], **59** (1996), no. 6, 865–880.
6. G. V. Chudnovsky, *Ann. of Math. (2)*, **117** (1983), no. 2, 325–382.
7. E. A. Rukhadze, *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* [*Moscow Univ. Math. Bull.*] (1987), no. 6, 25–29.
8. M. Hata, *J. Reine Angew. Math.*, **407** (1990), no. 1, 99–125.
9. G. Rhin and C. Viola, *Acta Arith.*, **77** (1996), no. 1, 23–56.
10. G. Rhin and C. Viola, *Acta Arith.*, **97** (2001), no. 3, 269–293.
11. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, *Encyclopedia of Mathematics and its Applications*, Cambridge Univ. Press, Cambridge, 1990.
12. P. Bundschuh and K. Väänänen, *Compositio Math.*, **91** (1994), 175–199.
13. Assche W. Van, *The Ramanujan J.*, **5** (2001), no. 3, 295–310.
14. V. V. Zudilin (W. Zudilin), *Manuscripta Math.*, **107** (2002), no. 4, 463–477.
15. E. Heine, *J. Reine Angew. Math.*, **34** (1847), 285–328.
16. V. V. Zudilin (W. Zudilin), E-print `math.NT/0206176` (August 2001).
17. V. V. Zudilin, *Mat. Sb.* [*Russian Acad. Sci. Sb. Math.*], **193** (2002), no. 8, 49–70.
18. Yu. V. Nesterenko, *Mat. Sb.* [*Russian Acad. Sci. Sb. Math.*], **187** (1996), no. 9, 65–96.
19. R. Apéry, *Astérisque*, **61** (1979), 11–13.
20. R. Askey, *Appl. Anal.*, **8** (1978), 125–141.
21. H. Exton, *q-Hypergeometric Functions and Applications*, Ellis Horwood Ser. Math. Appl., Ellis Horwood, Chichester, 1983.
22. F. Lindemann, *Math. Annalen*, **20** (1882), 213–225.
23. G. E. Andrews, R. Lewis, and Z.-G. Liu, *Bull. London Math. Soc.*, **33** (2001), 25–31.
24. T. Matala-aho and K. Väänänen, *Bull. Austral. Math. Soc.*, **58** (1998), 15–31.
25. M. Waldschmidt, *J. Théorie des nombres de Bordeaux*, **12** (2000), 581–595.

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