RAMANUJAN AND ODD ZETA VALUES

WADIM ZUDILIN

Ramanujan's vision of numbers was not directly linked with their arithmetic properties, so he never used the words "irrational" and "transcendental." Ramanujan thought about numbers from an analytical or — as we would word these days — computational perspective. He had a particular interest in efficient formulas for calculating some attractive mathematical quantities; π was his obvious favourite but the values of Riemann's zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

at integers $s=2,3,\dots$ did not escape his attention as well. Ramanujan clearly discovered on his own Euler's famous formula

$$\zeta(2k)=\frac{(-1)^{k-1}(2\pi)^k B_{2k}}{2\,(2k)!},\quad \text{where } k=1,2,\dots,$$

relating the even zeta values to the (rational) Bernoulli numbers. In a sharp contrast to this, no closed formula for any of the odd zeta values $\zeta(2k+1)$ for $k\ge 1$ is known; even the irrationality of the numbers remains open, with the sole exception of $\zeta(3)$ — this arithmetic result of Apéry (1978) stunned the mathematical world.

Below, in three parts, we highlight two concrete instances of Ramanujan's work involving odd zeta values, the instances that have had a tremendous impact on the theory of irrational and transcendental numbers.

1. Apéry's theorem

Ramanujan's notebooks reveal to us many examples of (irregular) continued fractions [5] including the following one for the Hurwitz zeta function at $s=3$ [3, Entry 32 (iii), p. 153]:

$$\sum_{n=1}^{\infty}\frac{1}{(n+x)^3}=\frac{1}{|P(0,x)}-\frac{1^6}{|P(1,x)}-\frac{2^6}{|P(2,x)}-\frac{3^6}{|P(3,x)}-\cdots \tag{1}$$

valid for $\operatorname{Re} x>-1/2$, where $P(n,x)=n^3+(n+1)^3+(4n+2)x(x+1)$. The discussion of this entry by Berndt in the second volume of Ramanujan's notebooks [3] was done ten years after Apéry had smashed the mathematics community by

Date: 31 January 2020.

2010 *Mathematics Subject Classification.* 11J72, 11M06, 11Y60, 33C20.

Key words and phrases. Ramanujan; zeta value; irrationality; modular form; hypergeometric series.

This text is intended for *The Ramanujan Encyclopedia*, K. Alladi, G. E. Andrews, B. C. Berndt and K. Ono (eds.).

his proof of the irrationality of $\zeta(3)$ (see [17]). Berndt has noted a striking connection between Apéry's approximations to the constant and the convergents to the continued fraction when $x = 1$. Shortly after that, F. Apéry [1] in a biographical note on his father R. Apéry stated that the construction in [2] was motivated from a "number table due to Ramanujan." This provides a sufficient ground to speculate about the actual involvement of Ramanujan in Apéry's proof. In the note [18], Rajkumar analyzes Apéry's method and concludes that the constructions in [2] are indeed based on Ramanujan's continued fraction (1).

No similar formula for another odd zeta value was given by Ramanujan. No formula of this type suitable for an irrationality proof of $\zeta(5)$ was ever discovered. But a development of the story after Apéry's discovery has finally led to some partial irrationality results for the odd zeta values. The reader is advised to follow the review [8] for an excellent account of this and, maybe, even to learn from [22] how to prove that at least one of $\zeta(5), \zeta(7), \dots, \zeta(s)$ is irrational for some odd s, by entirely elementary means. The most recent development about irrational odd zeta values can be found in [9, 14].

2. Ramanujan polynomials

In Ramanujan's notebooks, we find the following remarkable formula involving the odd zeta values [20, p. 173, Ch. 14, Entry 21 (i)], [21, pp. 319-320, formula (28)], [3, pp. 275-276]: For $\alpha, \beta > 0$ with $\alpha\beta = \pi^2$ and $k \in \mathbb{Z}$, $k > 0$,

$$\alpha^{-k}\left(\frac{1}{2}\zeta(2k+1) + \sum_{n=1}^{\infty}\frac{n^{-2k-1}}{e^{2\alpha n}-1}\right) = (-\beta)^{-k}\left(\frac{1}{2}\zeta(2k+1) + \sum_{n=1}^{\infty}\frac{n^{-2k-1}}{e^{2\beta n}-1}\right) - 2^{2k}\sum_{j=0}^{k+1}\frac{(-1)^j B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!}\alpha^{k+1-j}\beta^j, \qquad (2)$$

where, as before, B_{2k} are the Bernoulli numbers.

A rigorous proof of Ramanujan's formula together with a generalization were given by Grosswald [10] in 1970, and it is convenient to cast Ramanujan's formula (2) in the Eisenstein-series resembling form.

Theorem 1. *Set, for* $\operatorname{Im} z > 0$,

$$F_k(z) = \sum_{n=1}^{\infty}\frac{\sigma_k(n)}{n^k}e^{2\pi i n z}, \qquad \text{where} \quad \sigma_k(n) = \sum_{d\mid n} d^k,$$

and define the reciprocal Ramanujan *polynomial*

$$R_{2k+1}(z) = \sum_{j=0}^{k+1}\frac{B_{2j}B_{2k+2-2j}}{(2j)!(2k+2-2j)!}z^{2j} \in \mathbb{Q}[z].$$

Then

$$F_{2k+1}(z) - z^{2k}F_{2k+1}\left(-\frac{1}{z}\right) = \frac{\zeta(2k+1)}{2}(z^{2k}-1) + \frac{(2\pi i)^{2k+1}}{2z}R_{2k+1}(z).$$

The function $F_{2k+1}(z)$ is an example of an Eichler integral, and the above formula relates the values of two Eichler integrals to $\zeta(2k+1)$ through the Ramanujan polynomial. In particular, zeros of $R_{2k+1}(z)$ that lie in the upper half-plane and that are not $(2k)$th roots of unity give one a formula for $\zeta(2k+1)$ in terms of the Eichler integrals.

Theorem 2 (Ram Murty–Smyth–Wang (2011) [19]). *For each $k \ge 4$, there exists at least one algebraic number α with $\operatorname{Im}\alpha > 0$, $|\alpha| = 1$ and $\alpha^{2k} \ne 1$ such that $R_{2k+1}(\alpha) = 0$ and hence*

$$\frac{\zeta(2k+1)}{2} = \frac{F_{2k+1}(\alpha) - \alpha^{2k}F_{2k+1}(-1/\alpha)}{\alpha^{2k}-1}.$$

In other words, there exists an explicit formula for the Riemann zeta function at odd arguments $2k+1 \ge 9$ in terms of the difference of two Eichler integrals. On the other hand,

Theorem 3 (Gun–Ram Murty–Rath (2011) [11]). *The set*

$$\left\{\frac{F_{2k+1}(z) - z^{2k}F_{2k+1}(-1/z)}{z^{2k}-1} : \operatorname{Im} z > 0,\ z \in \overline{\mathbb{Q}},\ z^{2k} \ne 1\right\}$$

contains at most one algebraic number.

The Ramanujan polynomials are essential parts of the period polynomials for the classical Eisenstein series, and the distribution of their zeros is naturally linked to what is now called the Riemann Hypothesis for period polynomials [12]. The story above about the Ramanujan polynomials also motivated other mathematicians to consider similar polynomial families. The authors in [15] introduce five such related families, in which polynomials have coefficients which involve Bernoulli and Euler numbers, as well as odd zeta values. By applying theorems on polynomials due to Lakatos (2002), Schinzel (2005), and Lakatos and Losonczi (2009), as well as some inequalities for Bernoulli and Euler numbers, they have managed to show that in four of these families the polynomials have all their nontrivial zeros on the unit circle; the same result for the fifth family,

$$P_k(z) = \frac{(2\pi)^{2k-1}}{(2k)!}\sum_{j=0}^{k}(-1)^j B_{2j}B_{2k-2j}\binom{2k}{2j}z^{2j} + \zeta(2k-1)(z^{2k-1} + (-1)^k z),$$

was settled in the later work [16].

3. Generalized Lambert series

Ramanujan's formula (2) can be proved in several ways [4], and one of these approaches makes use of the connection with a Lambert series,

$$\sum_{n=1}^{\infty}\frac{n^{N-2h}}{e^{n^N x}-1},$$

the parameters N and h being integral, for $N=1$ and a transformation of the latter. The Lambert series in its generality was recorded by Ramanujan on p. 332 of his

Lost Notebook [21], however without a single hint at its possible transformations. This theme was first exploited by Kanemitsu, Tanigawa and Yoshimoto [13], where they proved a transformation in the case $0 < h \le N/2$, but it does not really settle the formula (2). The fact that this approach works without any restriction on the integral parameters N and h is substantiated in the recent work [7] of Dixit and Maji (see also the followup [6]).

Without attempting to give a review of the results in the papers [6, 7, 13] we indicate a particular generalization of (2), which can serve as a reasonable illustration to this production.

Theorem 4 (Dixit–Maji (2020) [7])**.** *Let N be an odd positive integer and $\alpha, \beta > 0$ such that $\alpha\beta^N = \pi^{N+1}$. Then for any non-zero integer m,*

$$\begin{aligned}
&\alpha^{-2Nk/(N+1)}\biggl(\frac{1}{2}\zeta(2Nk+1)+\sum_{n=1}^{\infty}\frac{n^{-2Nk-1}}{\exp(\alpha(2n)^N)-1}\biggr)\\
&= (-1)^k\beta^{-2Nk/(N+1)}\frac{2^{2k(N-1)}}{N}\biggl(\frac{1}{2}\zeta(2k+1)\\
&\qquad + (-1)^{(N-1)/2}\sum_{j=-(N-1)/2}^{(N-1)/2}(-1)^j\sum_{n=1}^{\infty}\frac{n^{-2k-1}}{\exp\bigl(\beta(2e^{\pi j\sqrt{-1}}n)^{1/N}\bigr)-1}\biggr)\\
&\qquad + (-1)^{k+(N-1)/2}2^{2Nk}\sum_{j=0}^{\lfloor k+(N+1)/(2N)\rfloor}\frac{(-1)^jB_{2j}B_{N+1+2N(k-j)}}{(2j)!\,(N+1+2N(k-j))!}\\
&\qquad\qquad \times \alpha^{2j/(N+1)}\beta^{N+2N^2(k-j)/(N+1)}.
\end{aligned}$$

References

[1] F. Apéry, Roger Apéry, 1916–1994: a radical mathematician, *Math. Intelligencer* **18**:2 (1996), 54–61.

[2] R. Apéry, Irrationalité de $\zeta(2)$ et $\zeta(3)$, *Astérisque* **61** (1979), 11–13.

[3] B. C. Berndt, *Ramanujan's Notebooks*, Part II (Springer-Verlag, New York, 1989).

[4] B. C. Berndt and A. Straub, Ramanujan's formula for $\zeta(2n+1)$, in "Exploring the Riemann zeta function", H. Montgomery, A. Nikeghbali and M. Rassias (eds.), (Springer, Cham, 2017), pp. 13–34.

[5] J. M. Borwein, A. van der Poorten, J. Shallit and W. Zudilin, *Neverending Fractions. An Introduction to Continued Fractions*, Austr. Math. Soc. Lecture Ser. **23** (Cambridge University Press, Cambridge, 2014).

[6] A. Dixit, R. Gupta, R. Kumar and B. Maji, Generalized Lambert series, Raabe's cosine transform and a generalization of Ramanujan's formula for $\zeta(2m+1)$, *Nagoya Math. J.* (to appear), 62 pp.; `doi:10.1017/nmj.2018.38`.

[7] A. Dixit and B. Maji, Generalized Lambert series and arithmetic nature of odd zeta values, Proc. Royal Soc. Edinburgh Sect. A Math. (to appear), 29 pp.; `doi:10.1017/prm.2018.146`.

[8] S. Fischler, Irrationalité de valeurs de zêta (d'après Apéry, Rivoal, ...), *Astérisque* **294** (2004), 27–62.

[9] S. Fischler, J. Sprang and W. Zudilin, Many odd zeta values are irrational, *Compos. Math.* **155** (2019), no. 5, 938–952.

[10] E. Grosswald, Die Werte der Riemannschen Zetafunktion an ungeraden Argumentstellen, *Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II* (1970), 9–13.

[11] S. Gun, M. Ram Murty and P. Rath, Transcendental values of certain Eichler integrals, *Bull. Lond. Math. Soc.* **43** (2011), no. 5, 939–952.

[12] S. Jin, W. Ma, K. Ono and K. Soundararajan, Riemann hypothesis for period polynomials of modular forms, *Proc. Nat. Acad. Sci USA* **113** (2016), no. 10, 2603–2608.

[13] S. Kanemitsu, Y. Tanigawa and M. Yoshimoto, On the values of the Riemann zeta-function at rational arguments, *Hardy–Ramanujan J.* **24** (2001), 11–19.

[14] L. Lai and P. Yu, A note on the number of irrational odd zeta values, *Compositio Math.* **156** (2020), no. 8, 1699–1717.

[15] M. N. Lalín and M. D. Rogers, Variations of the Ramanujan polynomials and remarks on $\zeta(2j+1)/\pi^{2j+1}$, *Funct. Approx. Comment. Math.* **48** (2013), part 1, 91–111.

[16] M. N. Lalín and C. J. Smyth, Unimodularity of zeros of self-inversive polynomials, *Acta Math. Hungar.* **138** (2013), no. 1-2, 85–101; Addendum, *Acta Math. Hungar.* **147** (2015), no. 1, 255–257.

[17] A. van der Poorten, A proof that Euler missed… Apéry's proof of the irrationality of $\zeta(3)$ (An informal report), *Math. Intelligencer* **1** (1978/79), no. 4, 195–203.

[18] K. Rajkumar, A simplification of Apéry's proof of the irrationality of $\zeta(3)$, *Preprint* `arχiv: 1212.5881 [math.NT]` (2012), 7 pp.

[19] M. Ram Murty, C. Smyth and R. J. Wang, Zeros of Ramanujan polynomials, *J. Ramanujan Math. Soc.* **26** (2011), no. 1, 107–125.

[20] S. Ramanujan, *Notebooks*, 2 volumes (Tata Institute of Fundamental Research, Bombay, 1957); 2nd ed. (2012).

[21] S. Ramanujan, *The Lost Notebook and other unpublished papers* (Narosa, New Delhi, 1988).

[22] W. Zudilin, One of the odd zeta values from $\zeta(5)$ to $\zeta(25)$ is irrational. By elementary means, *SIGMA Symmetry Integrability Geom. Methods Appl.* **14** (2018), paper no. 028, 8 pp.

Department of Mathematics, IMAPP, Radboud University, PO Box 9010, 6500 GL Nijmegen, Netherlands

E-mail address: `w.zudilin@math.ru.nl`