Izvestiya: Mathematics **66**:3 489–542

Izvestiya RAN: Ser. Mat. **66**:3 49–102
DOI 10.1070/IM2002v066n03ABEH000387

Irrationality of values of the Riemann zeta function

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Abstract. The paper deals with a generalization of Rivoal's construction, which enables one to construct linear approximating forms in 1 and the values of the zeta function $\zeta(s)$ only at odd points. We prove theorems on the irrationality of the number $\zeta(s)$ for some odd integers s in a given segment of the set of positive integers. Using certain refined arithmetical estimates, we strengthen Rivoal's original results on the linear independence of the $\zeta(s)$.

Introduction

The study of the arithmetical nature of the values of the Riemann zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}$$

at integers $s>1$ is one of the most attractive topics of the modern number theory. In spite of the deceptive simplicity of this problem, the results obtained in this area in the course of more than two centuries are far from being exhaustive. Euler's formula

$$\zeta(s)=-\frac{(2\pi i)^s B_s}{2s!},\qquad s=2,4,6,\dots,$$

in which the values of the zeta function at even integers are expressed in terms of $\pi\approx 3.1415926$ and the Bernoulli numbers $B_s\in\mathbb{Q}$, which are defined by the generating function

$$\frac{t}{e^t-1}=1-\frac{t}{2}+\sum_{s=2}^{\infty}B_s\frac{t^s}{s!},\qquad B_s=0\ \text{ for odd }\ s\geqslant 3,$$

undoubtedly marked the first progress in this area. In 1882 Lindemann proved that π is a transcendental number, which implies that $\zeta(s)$ is transcendental if s is even.

The problem of the irrationality of the values of the zeta function at odd integers seemed inaccessible until 1978, when Apéry [1] produced a sequence of rational approximations proving that $\zeta(3)$ is irrational.

This research was supported in part by INTAS and the Russian Foundation for Basic Research (grant no. IR-97-1904).

AMS 2000 *Mathematics Subject Classification.* 11J81, 11M06.

Apéry's Theorem. *The number $\zeta(3)$ is irrational.*

The history of this discovery and a rigorous mathematical justification of Apéry's observations can be found in [2]. The phenomenon of Apéry's sequence was repeatedly interpreted from the point of view of various analytical methods of number theory (see [3]–[8]). New approaches made it possible to strengthen Apéry's result *quantitatively*, that is, to obtain a "good" measure of the irrationality of $\zeta(3)$ ([9] and [10] represent the latest stages of the competition in this area).

Unfortunately, natural generalizations of Apéry's construction involve linear forms containing values of the zeta function at both odd and even integers (the interest in which faded after the Euler–Lindemann result). This circumstance makes it impossible to obtain results on the irrationality of $\zeta(s)$ for odd $s \geqslant 5$ by this method. Interesting attempts to approach this problem can be found in the preprints [11] and [12].

Finally, in 2000 Rivoal [13] equipped an auxiliary rational function with a symmetry and constructed linear forms containing the values of the zeta function only at odd integers $s > 1$.

Rivoal's Theorem. *The sequence $\zeta(3), \zeta(5), \zeta(7), \dots$ contains infinitely many irrational numbers. More precisely, the following estimate holds for the dimension $\delta(a)$ of the spaces generated over $\mathbb{Q}$ by $1, \zeta(3), \zeta(5), \dots, \zeta(a-2), \zeta(a)$ with an odd integer a:*

$$\delta(a) \geqslant \frac{\log a}{1 + \log 2}\bigl(1 + o(1)\bigr), \qquad a \to \infty.$$

In this paper we generalize Rivoal's construction [13] and prove the following theorems.

Theorem 0.1. *Each of the sets*

$$\begin{gathered} \{\zeta(5),\ \zeta(7),\ \zeta(9),\ \zeta(11),\ \zeta(13),\ \zeta(15),\ \zeta(17),\ \zeta(19),\ \zeta(21)\}, \\ \{\zeta(7),\ \zeta(9),\ \dots,\ \zeta(35),\ \zeta(37)\}, \qquad \{\zeta(9),\ \zeta(11),\ \dots,\ \zeta(51),\ \zeta(53)\} \end{gathered} \tag{0.1}$$

contains at least one irrational number.[1]

Theorem 0.2. *For every odd integer $b \geqslant 1$ at least one of the numbers*

$$\zeta(b+2),\ \zeta(b+4),\ \dots,\ \zeta(8b-3),\ \zeta(8b-1)$$

is irrational.

Theorem 0.3. *There are odd integers $a_1 \leqslant 145$ and $a_2 \leqslant 1971$ such that $1, \zeta(3), \zeta(a_1), \zeta(a_2)$ are linearly independent over $\mathbb{Q}$.*

Theorem 0.3 strengthens the corresponding theorem in [15], where it was established that $1, \zeta(3), \zeta(a)$ with some odd integer $a \leqslant 169$ are linearly independent.

[1]When the work on this paper was finished, I was informed that Rivoal [14] had obtained the assertion of Theorem 0.1 for the first set in (0.1) independently, using another generalization of the construction in [13].

Added in proof. I have recently obtained some refinements of Theorem 0.1. In particular, I have shown [28] that at least one of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.

Theorem 0.4. *The following absolute estimate holds for every odd integer $a \geqslant 3$:*

$$\delta(a) > 0.395 \log a > \frac{2}{3} \cdot \frac{\log a}{1+\log 2}. \tag{0.2}$$

This paper is organized as follows. In § 1 we describe an analytical construction of linear forms in the values of the zeta function at odd integers. In §§ 2–4 we study the asymptotics of linear forms, their coefficients and denominators. Finally, in § 5 we give complete proofs of the results. The main ingredient in the proofs of Theorems 0.3 and 0.4 (as well as in [13]) is the following special case of Nesterenko's theorem [16].

Criterion for linear independence. *Assume that for a given set of real numbers $\theta_0, \theta_1, \dots, \theta_m$, $m \geqslant 1$, one can find a sequence of linear forms*

$$I_n = A_{0,n}\theta_0 + A_{1,n}\theta_1 + \dots + A_{m,n}\theta_m, \qquad n = 1, 2, \dots,$$

with integer coefficients and numbers $\alpha > 0$ and $\beta > 0$ such that

$$\log |I_n| = -n\alpha + o(n), \qquad \log \max_{0\leqslant j\leqslant m} \bigl\{|A_{j,n}|\bigr\} \leqslant n\beta + o(n)$$

as $n \to \infty$. Then

$$\dim_{\mathbb{Q}}(\mathbb{Q}\theta_0 + \mathbb{Q}\theta_1 + \dots + \mathbb{Q}\theta_m) \geqslant 1 + \frac{\alpha}{\beta}.$$

Let us note that the justification of Theorems 0.1–0.4 is based on the saddle-point method and follows the scheme of proof of Apéry's theorem in [8]. It should be noted that some arithmetical results for the values of polylogarithms were obtained in [17], where the saddle-point method was used. The dissertation [17] served a link between our empirical observations and the rigorous justification in § 2. In § 4 we improve the arithmetical estimates (for the denominators of numerical linear forms) following [18], [9], [10], which enables us to refine the lower estimate for $\delta(a)$ in Theorems 0.3 and 0.4 for small values of a. Finally, in § 3 we obtain an upper estimate for the coefficients of linear forms and asymptotics of their growth.

The main results of this paper were announced in [19].

I am grateful to Professor Yu. V. Nesterenko for his constant interest and valuable advice, which have enabled me to improve this paper.

§ 1. An analytical construction

In this section we describe an analytical construction that enables us to obtain "good" linear forms with rational coefficients in the values of the zeta function at odd integers.

Let us fix positive integer parameters a, b, r such that $2rb \leqslant a$ and $a + b$ is even. For every positive integer n we consider the rational function

$$R(t) = R_n(t) := \frac{\bigl((t \pm (n+1)) \cdots (t \pm (n+2rn))\bigr)^b}{\bigl(t(t \pm 1) \cdots (t \pm n)\bigr)^a} \cdot (2n)!^{a-2rb}. \tag{1.1}$$

Here and below $(t \pm l)$ means that the product (sum or set) contains the factors (summands or elements) $t-l$ and $t+l$. We assign to the function defined by (1.1) the sum

$$I = I_n := \sum_{t=n+1}^{\infty} \frac{1}{(b-1)!} \frac{\mathrm{d}^{b-1} R(t)}{\mathrm{d}t^{b-1}}. \tag{1.2}$$

It is easy to see that the sum in (1.2) is taken only over integers $t > n + 2rn$. The series on the right-hand side of (1.2) converges absolutely, since the degrees of the numerator and the denominator of the rational function (1.1) are equal to $4rbn$ and $a(2n+1) \geqslant 4rbn + a \geqslant 4rbn + 2$, respectively, whence

$$R(t) = O\biggl(\frac{1}{t^2}\biggr) \quad \text{and} \quad \frac{\mathrm{d}^{b-1} R(t)}{\mathrm{d}t^{b-1}} = O\biggl(\frac{1}{t^2}\biggr), \qquad t \to \infty. \tag{1.3}$$

Applying Laplace's method to the integral representation of the sum in formula (1.2), we can calculate the quantity

$$\varkappa = \varlimsup_{n\to\infty} \frac{\log|I_n|}{n} > -\infty \tag{1.4}$$

(the details can be found in § 2 below), following [8] and [17].

Lemma 1.1. *For every* $n = 1, 2, \dots$ *the number defined by formula* (1.2) *is a linear form in* 1 *and the values of the zeta function at odd integers* s, $b < s < a + b$:

$$I = \sum_{\substack{s \text{ is odd} \\ b<s<a+b}} \bar{A}_s \zeta(s) - \bar{A}_0. \tag{1.5}$$

Proof. Decomposing the function (1.1) into a sum of partial fractions

$$R(t) = \sum_{k=-n}^{n} \biggl(\frac{A_{k,a}}{(t+k)^a} + \dots + \frac{A_{k,2}}{(t+k)^2} + \frac{A_{k,1}}{t+k}\biggr), \tag{1.6}$$

we note that

$$\sum_{k=-n}^{n} A_{k,1} = -\operatorname*{Res}_{t=\infty} R(t) = 0 \tag{1.7}$$

by (1.3). Since the function (1.1) is even (odd), that is,

$$R(-t) = (-1)^a R(t), \tag{1.8}$$

and the decomposition (1.6) is unique, we have

$$A_{k,j} = (-1)^{a-j} A_{-k,j}, \qquad k = 0, \pm 1, \dots, \pm n, \quad j = 1, 2, \dots, a.$$

Hence,

$$\sum_{k=-n}^{n} A_{k,j} = (-1)^{a-j} \sum_{k=-n}^{n} A_{k,j} = 0 \quad \text{if } a - j \text{ is odd}, \qquad j = 2, \dots, a. \tag{1.9}$$

Substituting (1.6) into (1.2), we obtain that

$$(-1)^{b-1}I=\sum_{t=n+1}^{\infty}\sum_{k=-n}^{n}\biggl(\binom{a+b-2}{b-1}\frac{A_{k,a}}{(t+k)^{a+b-1}}+\binom{a+b-3}{b-1}\frac{A_{k,a-1}}{(t+k)^{a+b-2}}+\cdots+\binom{b}{b-1}\frac{A_{k,2}}{(t+k)^{b+1}}+\frac{A_{k,1}}{(t+k)^{b}}\biggr).$$

Simple transformations yield the formula

$$I=\bar{A}_{a+b-1}\zeta(a+b-1)+\bar{A}_{a+b-2}\zeta(a+b-2)+\cdots+\bar{A}_{b}\zeta(b)-\bar{A}_{0}, \tag{1.10}$$

where

$$\bar{A}_j=(-1)^{b-1}\binom{j-1}{b-1}\sum_{k=-n}^{n}A_{k,j-b+1}, \qquad j=b,b+1,\dots,a+b-1, \tag{1.11}$$

$$\bar{A}_0=(-1)^{b-1}\sum_{k=-n}^{n}\sum_{l=1}^{k+n}\biggl(\binom{a+b-2}{b-1}\frac{A_{k,a}}{l^{a+b-1}}+\cdots+\binom{b}{b-1}\frac{A_{k,2}}{l^{b+1}}+\frac{A_{k,1}}{l^{b}}\biggr). \tag{1.12}$$

By (1.7), we have $\bar{A}_b=0$. Hence, the expression on the right-hand side of (1.10) is well defined even in the case when $b=1$. Since $a+b$ is even, formula (1.9) implies that $\bar{A}_j=0$ if $j\geqslant b$ is even. Hence, formula (1.10) means that I is a linear form in 1 and the values of the zeta function at odd integers s, $b+1\leqslant s\leqslant a+b-1$, as was to be shown.

The following lemma enables us to calculate the denominators of linear forms (1.2).

Lemma 1.2 (cf. [20], [13]). *Assume that for some polynomial $P(t)$ of degree $\leqslant n$ the rational function*

$$R(t)=\frac{P(t)}{(t+s)(t+s+1)\cdots(t+s+n)} \tag{1.13}$$

(*this representation does not have to be irreducible*) *satisfies the conditions*

$$\bigl(R(t)(t+k)\bigr)\big|_{t=-k}\in\mathbb{Z}, \qquad k=s,s+1,\dots,s+n. \tag{1.14}$$

Then

$$\frac{D_n^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\bigl(R(t)(t+k)\bigr)\bigg|_{t=-k}\in\mathbb{Z}, \qquad k=s,s+1,\dots,s+n,$$

for all non-negative integers j, where D_n is the least common multiple of $1,2,\dots,n$.

Proof. Decomposing the rational function (1.13) into a sum of partial fractions, we obtain that

$$R(t)=\sum_{l=s}^{s+n}\frac{B_l}{t+l},$$

where

$$B_l = \big(R(t)(t+l)\big)\big|_{t=-l} \in \mathbb{Z}, \qquad l = s, s+1, \dots, s+n.$$

Therefore,

$$R(t)(t+k) = B_k + \sum_{\substack{l=s\\ l\neq k}}^{s+n} B_l \frac{t+k}{t+l} = B_k + \sum_{\substack{l=s\\ l\neq k}}^{s+n} B_l \left(1 + \frac{k-l}{t+l}\right),$$

whence

$$\frac{1}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\big(R(t)(t+k)\big)\bigg|_{t=-k} = (-1)^j \sum_{\substack{l=s\\ l\neq k}}^{s+n} \frac{B_l(k-l)}{(t+l)^{j+1}}\bigg|_{t=-k} = -\sum_{\substack{l=s\\ l\neq k}}^{s+n} \frac{B_l}{(k-l)^j}.$$

This completes the proof of the assertion.

Remark 1.1. Lemma 1.2 was stated by Nesterenko [21]. Nikishin [20] applied it to the polynomial

$$P(t) = (t+p+1)\cdots(t+p+n)$$

with $p - s \in \mathbb{Z}$ and $p + n < s$ or $p \geqslant s + n$. Rivoal observed [13] that (1.14) holds for $P(t) = n!$. In each of these cases B_k, $k = s, s+1, \dots, s+n$, are binomial coefficients, and (1.14) is easily verified.

It is easy to see that the rational function (1.1) can be written as a product of the functions

$$F(t) = F_{n,m}(t) := \frac{(t \pm (m+1))\cdots(t \pm (m+2n))}{\big(t(t\pm 1)\cdots(t\pm n)\big)^2}, \qquad n \leqslant m, \tag{1.15}$$

$$H(t) = H_n(t) := \frac{(2n)!}{t(t\pm 1)\cdots(t\pm n)} \tag{1.16}$$

(see formula (1.22) below).

Lemma 1.3. *The following inclusions hold for functions* (1.15) *and* (1.16):

$$\frac{D_{2n}^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\big(F(t)(t+k)^2\big)\bigg|_{t=-k} \in \mathbb{Z}, \qquad k = 0, \pm 1, \dots, \pm n, \quad j = 0, 1, 2, \dots, \tag{1.17}$$

$$\frac{D_{2n}^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\big(H(t)(t+k)\big)\bigg|_{t=-k} \in \mathbb{Z}, \qquad k = 0, \pm 1, \dots, \pm n, \quad j = 0, 1, 2, \dots, \tag{1.18}$$

where D_{2n} *is the least common multiple of* $1, 2, \dots, 2n$.

Proof. The inclusions (1.18) follow from Lemma 1.2 applied to the rational function (1.16). We prove (1.17) using Lemma 1.2 for the rational functions

$$\begin{aligned} R_-(t) &= \frac{(t-(m+1))\cdots(t-(m+2n))}{t(t\pm 1)\cdots(t\pm n)}, \\ R_+(t) &= \frac{(t+(m+1))\cdots(t+(m+2n))}{t(t\pm 1)\cdots(t\pm n)} \end{aligned} \tag{1.19}$$

and the Leibniz rule for the jth derivative of the product $R_-(t)R_+(t) = F(t)$. The assumptions (1.14) hold for each rational function in (1.16) and (1.19), since

$$\big(H(t)(t+k)\big)\big|_{t=-k} = (-1)^{n+k}\frac{(2n)!}{(n+k)!\,(n-k)!} = (-1)^{n+k}\binom{2n}{n+k} \in \mathbb{Z},$$

$$\big(R_-(t)(t+k)\big)\big|_{t=-k} = \binom{m+2n+k}{2n}\cdot(-1)^{n+k}\binom{2n}{n+k} \in \mathbb{Z},$$

$$\big(R_+(t)(t+k)\big)\big|_{t=-k} = \binom{m+2n-k}{2n}\cdot(-1)^{n+k}\binom{2n}{n+k} \in \mathbb{Z}$$

for all $k = 0, \pm1, \dots, \pm n$. This completes the proof of the lemma.

In what follows the *denominator* $\operatorname{den}(I)$ of the linear form $I = A_0 + A_1\theta_1 + \dots + A_m\theta_m$ is defined to be the least *rational* (not necessarily integer) $D > 0$ such that every $DA_0, DA_1, \dots, DA_m$ is an integer.

Lemma 1.4. *The following inclusions hold for the coefficients of the linear form* (1.5):

$$D_{2n}^{a+b-1-s}\bar{A}_s \in \mathbb{Z}, \tag{1.20}$$

where s is zero or an odd integer, $b < s < a+b$. Therefore, $\operatorname{den}(I_n)$ *divides* D_{2n}^{a+b-1} *and*

$$\varlimsup_{n\to\infty}\frac{\log\operatorname{den}(I_n)}{n} \leqslant \lim_{n\to\infty}\frac{\log D_{2n}^{a+b-1}}{n} = 2(a+b-1). \tag{1.21}$$

Proof. To determine the coefficients in (1.6), we use the formula

$$A_{k,j} = \frac{1}{(a-j)!}\,\frac{\mathrm{d}^{a-j}}{\mathrm{d}t^{a-j}}\big(R(t)(t+k)^a\big)\bigg|_{t=-k}, \qquad k = 0, \pm1, \dots, \pm n, \quad j = 1, 2, \dots, a,$$

and the representation

$$R(t) = (H_n(t))^{a-2rb}\,(F_{n,n}(t))^b\,(F_{n,3n}(t))^b \cdots (F_{(2r-1)n,n}(t))^b \tag{1.22}$$

of the rational function (1.1). The inclusions

$$D_{2n}^{a-j}A_{k,j} \in \mathbb{Z}, \qquad k = 0, \pm1, \dots, \pm n, \quad j = 1, 2, \dots, a, \tag{1.23}$$

follow from Lemma 1.3 and the Leibniz rule for the differentiation of a product. Inclusions (1.20) follow from (1.11), (1.12) and (1.23). The limiting relation (1.21) follows from the formula

$$\lim_{n\to\infty}\frac{\log D_n}{n} = 1 \tag{1.24}$$

(the prime number theorem). This completes the proof of the lemma.

So we have constructed a sequence of linear forms I_n, $n = 1, 2, \dots$. By (1.4), it contains an infinite subsequence of non-zero forms. The forms $D_{2n}^{a+b-1}I_n$, $n = 1, 2, \dots$, have integer coefficients. Hence, for

$$\varkappa + 2(a+b-1) < 0 \tag{1.25}$$

at least one of the numbers $\zeta(s)$, where s is an odd integer and $b < s < a+b$, is irrational. Moreover, upper estimates for the coefficients of the linear forms (1.10) in the case when (1.25) holds enable us to obtain a lower estimate for the number of irrational numbers in the given set of values of the zeta function. This is why our next step will be to calculate (1.4) and to find the asymptotics of the coefficients of the linear forms (1.10) as $n \to \infty$.

§ 2. An asymptotic estimate for linear forms

We shall need some supplementary properties of the function $\cot z$.

Lemma 2.1. *The maximum value of the real-valued non-negative function* $h(y) = |\cot(x+iy)|$, $y \in \mathbb{R}$, *depends only on* $x \pmod{\pi\mathbb{Z}} \in \mathbb{R}$.

Proof. If $x = \pi k$ for some integer k, then $z = x$ is a (unique) pole of the function $\cot z$ on the line $\operatorname{Im} z = x$ (see (2.6) below). Therefore, in this case the absolute maximum of $h(y)$, which is equal to infinity, is attained at $y = 0$. In what follows we assume that $x \notin \pi\mathbb{Z}$, whence $\cos 2x < 1$. We have

$$\begin{aligned} h(y)^2 &= \left|\frac{\cos(x+iy)}{\sin(x+iy)}\right|^2 = \left|\frac{(e^{-y}+e^{y})\cos x + i(e^{-y}-e^{y})\sin x}{(e^{-y}-e^{y})\cos x + i(e^{-y}+e^{y})\sin x}\right|^2 \\ &= \frac{e^{-2y}+e^{2y}+2\cos 2x}{e^{-2y}+e^{2y}-2\cos 2x} = 1 + \frac{4\cos 2x}{e^{-2y}+e^{2y}-2\cos 2x} \\ &\leqslant 1 + \frac{4|\cos 2x|}{2-2\cos 2x}. \end{aligned} \tag{2.1}$$

Here we have used the inequality $Y + 1/Y \geqslant 2$ for $Y = e^{-2y} > 0$, which becomes an equality only in the case when $Y = 1$. The estimate on the right-hand side of (2.1) depends only on $x \pmod{\pi\mathbb{Z}}$, which completes the proof of the lemma.

We define "differential iterations" of the cotangent:

$$\cot_b z = \frac{(-1)^{b-1}}{(b-1)!}\,\frac{\mathrm{d}^{b-1}\cot z}{\mathrm{d}z^{b-1}}, \qquad b = 1, 2, \dots. \tag{2.2}$$

Lemma 2.2. *For every* $b = 1, 2, \dots$

(a) *the function* $\cot_b z$ *is a polynomial in* $\cot z$ *with rational coefficients*:

$$\cot_b z = U_b(\cot z), \qquad U_b(-y) = (-1)^b U_b(y), \quad \deg U_b = b;$$

(b) *the function* $\sin^b z \cdot \cot_b z$ *is a polynomial in* $\cos z$ *with rational coefficients*:

$$\sin^b z \cdot \cot_b z = V_b(\cos z), \qquad V_b(-y) = (-1)^b V_b(y), \quad \deg V_b = \max\{1, b-2\}. \tag{2.3}$$

Proof. We proceed by induction. For $b = 1$ we have $\cot_1 z = \cot z$ and $\sin z \cdot \cot_1 z = \cos z$, whence $U_1(y) = V_1(y) = y$.

(a) According to the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}z} = -(y^2+1), \qquad y = \cot z,$$

and formula (2.2), the following recurrence relation holds for the polynomials $U_b(y)$, $b = 1, 2, \dots$:

$$U_{b+1}(y) = \frac{1}{b}(y^2+1)U_b'(y), \qquad b = 2, 3, \dots.$$

The induction step from b to $b+1$ shows that $U_{b+1}(-y) = (-1)^{b+1}U_{b+1}(y)$ and the degree of U_{b+1} is one greater than the degree of U_b.

(b) Assume that (2.3) holds for some integer $b \geqslant 1$. Then

$$\begin{aligned}\sin z \cdot \frac{\mathrm{d}}{\mathrm{d}z} V_b(\cos z) &= \sin z \cdot \frac{\mathrm{d}}{\mathrm{d}z}(\sin^b z \cdot \cot_b z) \\ &= b \cos z \cdot \sin^b z \cdot \cot_b z + \sin^{b+1} z \cdot \frac{\mathrm{d}}{\mathrm{d}z} \cot_b z \\ &= b \cos z \cdot V_b(\cos z) - b \sin^{b+1} z \cdot \cot_{b+1} z,\end{aligned}$$

whence

$$\begin{aligned}\sin^{b+1} z \cdot \cot_{b+1} z &= \cos z \cdot V_b(\cos z) - \frac{1}{b} \sin z \cdot \frac{\mathrm{d}}{\mathrm{d}z} V_b(\cos z) \\ &= \cos z \cdot V_b(\cos z) + \frac{1}{b} \sin^2 z \cdot V_b'(\cos z) \\ &= V_{b+1}(\cos z),\end{aligned}$$

where

$$V_{b+1}(y) = y V_b(y) + \frac{1}{b}(1 - y^2) V_b'(y). \tag{2.4}$$

Formula (2.4) and the induction hypothesis imply that $V_{b+1}(y)$ is a polynomial whose degree cannot exceed the degree of $V_b(y)$ more than by one. Besides, $U_{b+1}(-y) = (-1)^{b+1} U_{b+1}(y)$. Since $V_2(y) = 1$ (by (2.4)), we have $\deg V_{b+1} \leqslant b - 1$. The induction step from b to $b+1$ and relation (2.4) show that the coefficient of y^{b-1} in the polynomial $V_{b+1}(y)$ is non-zero and equals $2^{b-1}/b!$. Hence, (2.3) holds for all $b = 1, 2, \ldots$, which completes the proof of the lemma.

Lemma 2.3. *For any $b = 1, 2, \ldots$ and any integer k the following representation holds in the neighbourhood of $t = k$:*

$$\pi^b \cot_b \pi t = \frac{1}{(t-k)^b} + O(1), \tag{2.5}$$

where the function in $O(1)$ is analytic in the neighbourhood of $t = k$. The function $\cot_b \pi t$ *is analytic in the neighbourhood of $t \in \mathbb{C} \setminus \mathbb{Z}$*

Proof. The second assertion of the lemma follows from formula (2.2) defining the function $\cot_b z$ and the decomposition of the cotangent into a sum of partial fractions:

$$\pi \cot \pi t = \frac{1}{t} + \sum_{m=1}^{\infty} \left(\frac{1}{t-m} + \frac{1}{t+m} \right) \tag{2.6}$$

(see, for example, [22], Ch. X, § 3, formula (3)).

In the neighbourhood of $t = k \in \mathbb{Z}$ we have

$$\pi \cot \pi t = \pi \cot \pi (t-k) = \frac{1}{t-k} \cdot \left(1 + \sum_{s=2}^{\infty} B_s \frac{(2\pi i)^s (t-k)^s}{s!} \right), \tag{2.7}$$

where B_s, $s = 2, 3, \ldots$, are the Bernoulli numbers (see, for example, [22], Ch. X, § 3, formula (2)). Differentiating identity (2.7) $b - 1$ times, we obtain representation (2.5) and the first assertion of the lemma, which completes the proof.

Lemma 2.4. *The following integral representation holds for the sum in* (1.2):

$$I = \frac{\pi^{b-1} i}{2} \int_{M-i\infty}^{M+i\infty} \cot_b \pi t \cdot R(t)\,\mathrm{d}t, \tag{2.8}$$

where $M \in \mathbb{R}$ is an arbitrary constant in the interval $n < M < (2r+1)n$.

Proof. Consider the integrand in (2.8) on the contour of the rectangle with vertices $M \pm iN$, $N + 1/2 \pm iN$ (see Fig. 1), where the integer N is sufficiently large, $N > (2r+1)n$. Since the function $R(t)$ is analytic inside this rectangle and on its boundary, Cauchy's theorem and Lemma 2.3 imply that the integral

$$\frac{\pi^b}{2\pi i}\biggl(\int_{M+iN}^{M-iN} + \int_{M-iN}^{N+1/2-iN} + \int_{N+1/2-iN}^{N+1/2+iN} + \int_{N+1/2+iN}^{M+iN}\biggr) \cot_b \pi t \cdot R(t)\,\mathrm{d}t \tag{2.9}$$

is equal to the sum of the residues of the integrand at $t \in \mathbb{Z}$, $M < t \leqslant N$. Using (2.5) and the expansion

$$R(t) = R(k) + R'(k)\cdot(t-k) + \dots + \frac{R^{(b-1)}(k)}{(b-1)!}\cdot(t-k)^{b-1} + O\bigl((t-k)^b\bigr)$$

in the neighbourhood of $t = k \in \mathbb{Z}$, $M < k \leqslant N$, we obtain that the integral (2.9) is equal to

$$\sum_{M<k\leqslant N} \operatorname*{Res}_{t=k}\bigl(\pi^b \cot_b \pi t \cdot R(t)\bigr) = \sum_{M<k\leqslant N} \frac{R^{(b-1)}(k)}{(b-1)!} = \sum_{k=n+1}^{N} \frac{R^{(b-1)}(k)}{(b-1)!}. \tag{2.10}$$

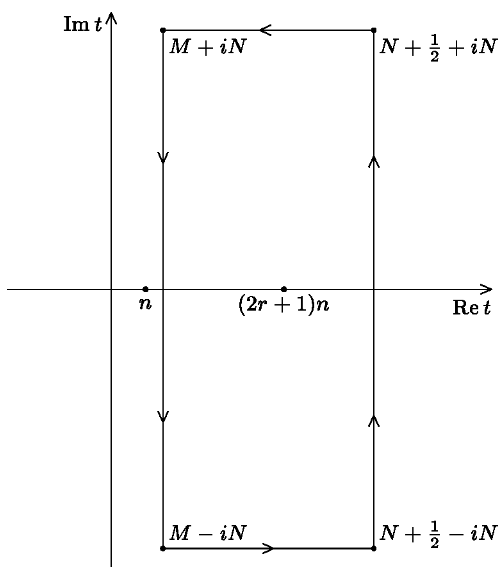

Figure 1. The neighbourhood of the point x_*

Let us note that on the sides $[N+1/2-iN, N+1/2+iN]$, $[M-iN, N+1/2-iN]$ and $[N+1/2+iN, M+iN]$ of the rectangle we have $R(t)=O(N^{-2})$ by (1.3), and the function $\cot_b \pi t$ is bounded. The latter assertion follows from assertion (a) of Lemma 2.2 and the boundedness of $\cot \pi t$ on the sides under consideration: for the segment $[N+1/2-iN, N+1/2+iN]$ we use Lemma 2.1, while for the other two segments we use the trivial estimate

$$|\cot \pi(x+iy)| = \left|\frac{1+e^{\pm 2\pi ix}\cdot e^{-2\pi|y|}}{1-e^{\pm 2\pi ix}\cdot e^{-2\pi|y|}}\right| \leqslant \frac{1+e^{-2\pi|y|}}{1-e^{-2\pi|y|}}, \qquad x\in\mathbb{R},$$

for $y=\pm N$. Therefore,

$$\left(\int_{M-iN}^{N+1/2-iN} + \int_{N+1/2-iN}^{N+1/2+iN} + \int_{N+1/2+iN}^{M+iN}\right) \cot_b \pi t \cdot R(t)\,\mathrm{d}t = O(N^{-1}).$$

Passing to the limit in (2.9) as $N\to\infty$, we obtain the right-hand side of (2.8). Passing to the limit in (2.10), we obtain the desired sum (1.2), which completes the proof of the lemma.

Lemma 2.5. *The following relation holds for* (1.2) *as* $n\to\infty$:

$$I = \tilde{I}\,\frac{(-1)^{bn}(2\sqrt{\pi n}\,)^{a-2rb}(2\pi)^b}{n^{a-1}}\bigl(1+O(n^{-1})\bigr),$$

where

$$\tilde{I} = \tilde{I}_n := -\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} \sin^b \pi n\tau \cdot \cot_b \pi n\tau \cdot e^{nf(\tau)}\cdot g(\tau)\,\mathrm{d}\tau, \tag{2.11}$$

$$\begin{aligned} f(\tau) &= b(\tau+2r+1)\log(\tau+2r+1) + b(-\tau+2r+1)\log(-\tau+2r+1) \\ &\quad + (a+b)(\tau-1)\log(\tau-1) - (a+b)(\tau+1)\log(\tau+1) + (a-2rb)2\log 2, \end{aligned} \tag{2.12}$$

$$g(\tau) = \frac{(\tau+2r+1)^{b/2}(-\tau+2r+1)^{b/2}}{(\tau+1)^{(a+b)/2}(\tau-1)^{(a+b)/2}}, \tag{2.13}$$

and $\mu\in\mathbb{R}$ *is an arbitrary constant in the interval* $1<\mu<2r+1$.

Remark 2.1. To define the functions $f(\tau)$ and $g(\tau)$ unambiguously, we consider them in the τ-plane with cuts along the rays $(-\infty,1]$ and $[2r+1,+\infty)$, fixing the branches of logarithms that take real values on the interval $(1,2r+1)$ of the real axis. This choice of branches is motivated by the following proof.

Proof. Using the formula $\Gamma(z+1)=z\Gamma(z)$, we express the rational function (1.1) in terms of the gamma function:

$$R(t) = (-1)^{an}\left(\frac{\Gamma(\pm t+(2r+1)n+1)}{\Gamma(\pm t+n+1)}\right)^b \left(\frac{\Gamma(t)\Gamma(1-t)}{\Gamma(\pm t+n+1)}\right)^a (2n)!^{a-2rb}, \tag{2.14}$$

where, as before, $\pm$ stands for two factors in the product. We transform the integrand in (2.8) using formula (2.14) for $R(t)$ and the identity

$$\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin \pi t} = (-1)^n\Gamma(-t+n+1)\Gamma(t-n).$$

We have

$$R(t) = \frac{(-1)^{bn} \cdot \sin^b \pi t}{\pi^b} \, \frac{\Gamma(\pm t + (2r+1)n + 1)^b \Gamma(t-n)^{a+b}}{\Gamma(t+n+1)^{a+b}} \, (2n)!^{a-2rb}. \tag{2.15}$$

Substituting this expression into (2.8), we obtain the formula

$$\begin{aligned} I &= \frac{(-1)^{bn} i}{2\pi} \int_{M-i\infty}^{M+i\infty} \sin^b \pi t \cdot \cot_b \pi t \\ &\quad \times \frac{\Gamma(\pm t + (2r+1)n + 1)^b \Gamma(t-n)^{a+b} (2n)!^{a-2rb}}{\Gamma(t+n+1)^{a+b}} \, \mathrm{d}t \\ &= \frac{(-1)^{bn} (2n)^{a-2rb} i}{2\pi} \int_{M-i\infty}^{M+i\infty} \sin^b \pi t \cdot \cot_b \pi t \cdot \frac{(-t + (2r+1)n)^b (t + (2r+1)n)^b}{(t+n)^{a+b}} \\ &\quad \times \frac{\Gamma(\pm t + (2r+1)n)^b \Gamma(t-n)^{a+b} \Gamma(2n)^{a-2rb}}{\Gamma(t+n)^{a+b}} \, \mathrm{d}t, \end{aligned} \tag{2.16}$$

where $M \in \mathbb{R}$ is an arbitrary constant in the interval $n < M < (2r+1)n$. We put $M = \mu n$ for $\mu \in \mathbb{R}$ in the interval $1 < \mu < 2r+1$.

The asymptotics of the gamma function

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + \log \sqrt{2\pi} + O\big(|z|^{-1}\big) \tag{2.17}$$

(see, for example, [23], § 3.10) implies that the following formulae hold on the contour of integration in (2.16):

$$\begin{aligned} \log \Gamma(\pm t + (2r+1)n) &= \left(\pm t + (2r+1)n - \frac{1}{2}\right) \log(\pm t + (2r+1)n) \\ &\qquad - (\pm t + (2r+1)n) + \log \sqrt{2\pi} + O(n^{-1}), \\ \log \Gamma(t+n) &= \left(t + n - \frac{1}{2}\right) \log(t+n) - (t+n) + \log \sqrt{2\pi} + O(n^{-1}), \\ \log \Gamma(t-n) &= \left(t - n - \frac{1}{2}\right) \log(t-n) - (t-n) + \log \sqrt{2\pi} + O(n^{-1}), \\ \log \Gamma(2n) &= \left(2n - \frac{1}{2}\right) \log(2n) - 2n + \log \sqrt{2\pi} + O(n^{-1}). \end{aligned}$$

The change $t = n\tau$ in the integrals in (2.16) yields

$$\begin{aligned} I = \frac{(-1)^{bn} (2\sqrt{\pi n}\,)^{a-2rb} (2\pi)^{b-1} i}{n^{a-1}} &\int_{\mu-i\infty}^{\mu+i\infty} \sin^b \pi n\tau \cdot \cot_b \pi n\tau \cdot e^{nf(\tau)} \\ \times \frac{(\tau + 2r + 1)^{b/2} (-\tau + 2r + 1)^{b/2}}{(\tau+1)^{(a+b)/2} (\tau-1)^{(a+b)/2}} &\big(1 + O(n^{-1})\big) \, \mathrm{d}\tau, \end{aligned}$$

where $f(\tau)$ is the function defined in (2.12) and $\mu = M/n \in \mathbb{R}$ is an arbitrary constant in the interval $1 < \mu < 2r+1$. This completes the proof of the lemma.

By assertion (b) of Lemma 2.2 the integral in (2.11) can be written as

$$\tilde{I} = -\frac{1}{2\pi i}\sum_{k=-b}^{b} c_k \int_{\mu-i\infty}^{\mu+i\infty} e^{n(f(\tau)-k\pi i\tau)} g(\tau)\,\mathrm{d}\tau, \qquad c_k = c_{-k}, \tag{2.18}$$

where the sum is taken over those k that have even parity with b, and $c_{-b} = c_b = 0$ if $b > 1$.

Lemma 2.6. *For $\mu, \lambda \in \mathbb{R}$, $1 < \mu < 2r+1$, and n a positive integer we put*

$$J_{n,\lambda} = \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} e^{n(f(\tau)-\lambda\pi i\tau)} g(\tau)\,\mathrm{d}\tau. \tag{2.19}$$

Then

$$J_{n,-\lambda} = \overline{J_{n,\lambda}},$$

where the bar stands for complex conjugation.

Proof. Applying Schwarz' reflection principle to the functions $f(\tau), g(\tau)$ analytic in $\mathbb{C} \setminus ((-\infty, 1] \cup [2r+1, +\infty))$ and taking real values on the interval $(1, 2r+1)$ (see, for example, [24], Ch. III, § 7, Proposition 7.1) and making the change $\tau \mapsto \bar{\tau}$ in the integral in (2.19), we obtain that

$$\begin{aligned} J_{n,\lambda} &= \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} e^{n(f(\bar\tau)-\lambda\pi i\bar\tau)} g(\bar\tau)\,\mathrm{d}\bar\tau = -\frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty} e^{n(f(\bar\tau)-\lambda\pi i\bar\tau)} g(\bar\tau)\,\mathrm{d}\tau \\ &= \overline{\frac{1}{2\pi i}}\int_{\mu-i\infty}^{\mu+i\infty} e^{n(\overline{f(\tau)+\lambda\pi i\tau})}\,\overline{g(\tau)}\,\mathrm{d}\tau = \overline{J_{n,-\lambda}}, \end{aligned}$$

as was to be shown.

Corollary 2.1. *The integral in* (2.11) *can be written as*

$$\tilde{I} = -2\sum_{\substack{k=0\\ k\equiv b\ (\mathrm{mod}\ 2)}}^{b} c_k \operatorname{Re} J_{n,k}, \tag{2.20}$$

where c_k are some (rational) constants, $c_1 = 1$ for $b = 1$ and $c_b = 0$, $c_{b-2} \neq 0$ for $b > 1$.

Proof. The desired representation (2.20) follows immediately from assertion (b) of Lemma 2.2, formula (2.18), and Lemma 2.6.

Now let us calculate the asymptotics of the integrals on the right-hand side of (2.18) using the saddle-point method. We have to determine the saddle points of the integrands, that is, the zeros of the derivatives of the functions $f(\tau) - k\pi i\tau$, $k = 0, \pm1, \dots, \pm b$, $k \equiv b \pmod 2$. It is easy to verify that the roots of the derivative

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}(f(\tau) - k\pi i\tau) &= b\log(\tau+2r+1) - b\log(-\tau+2r+1) \\ &\quad + (a+b)\log(\tau-1) - (a+b)\log(\tau+1) - k\pi i, \\ k &= 0, \pm1, \dots, \pm b, \quad k \equiv b \pmod 2, \end{aligned}$$

are simultaneously the roots of the polynomial

$$(\tau+2r+1)^b(\tau-1)^{a+b}-(\tau-2r-1)^b(\tau+1)^{a+b}, \tag{2.21}$$

whose quantity is equal to the degree $a+2b-1$ of this polynomial.

Lemma 2.7. *Let a,b,r be positive integers, let $a+b$ be even, $a\geqslant 3rb$, and let $f(\tau)$ be the function defined by* (2.12), *where the corresponding branches of the logarithms in the cut plane $\mathbb{C}\setminus((-\infty,1]\cup[2r+1,+\infty))$ take real values on the interval* $(1,2r+1)$. *Then the equation*

$$f'(\tau)-\lambda\pi i=0, \qquad \lambda\in\mathbb{R}, \tag{2.22}$$

has

(a) *a pair of real solutions $-\mu_0\pm i0$ with μ_0 symmetric with respect to the imaginary axis for $\lambda=0$, where $\mu_0\in(1,2r+1)$, and $+$ $(-)$ in $\pm i0$ corresponds to the upper* (*lower*) *bank of the cut* $(-\infty,1]$,

(b) *a pair of real solutions $-\mu_1\pm i0$ and $\mu_1\pm i0$ symmetric with respect to the imaginary axis for $\lambda=\pm b$, where $\mu_1\in(2r+1,+\infty)$, and $+$ $(-)$ in $\pm i0$ coincides with the sign of λ and corresponds to the upper* (*lower*) *banks of the cuts* $(-\infty,1]$, $[2r+1,+\infty)$,

(c) *a real solution $\pm i0$ for $\lambda=\pm(a+b)$, where $+$ $(-)$ in $\pm i0$ coincides with the sign of λ and corresponds to the upper* (*lower*) *bank of the cut* $(-\infty,1]$,

(d) *a solution on the imaginary axis for the real λ such that $b<|\lambda|<a+b$,*

(e) *a pair of complex solutions symmetric with respect to the imaginary axis for the real λ such that $0<|\lambda|<b$.*

All solutions of equation (2.22) *corresponding to positive λ are contained in the half-plane* $\operatorname{Im}\tau>0$. *All solutions corresponding to negative λ are contained in the half-plane* $\operatorname{Im}\tau<0$. *All solutions of equation* (2.22) *appear in the list* (a)–(e).

Proof. Since

$$\begin{aligned} f'(\tau)=b\log(\tau+2r+1)-b\log(-\tau+2r+1)\\ +(a+b)\log(\tau-1)-(a+b)\log(\tau+1) \end{aligned} \tag{2.23}$$

and $\log z=\log|z|+i\arg z$, $-\pi<\arg z<\pi$, for the main branch of the logarithm with the cut $(-\infty,0]$ in the z-plane, we have

$$\operatorname{Re}f'(\tau)=\log\frac{|\tau+2r+1|^b|\tau-1|^{a+b}}{|-\tau+2r+1|^b|\tau+1|^{a+b}}, \tag{2.24}$$

$$\begin{aligned} \operatorname{Im}f'(\tau)=b\arg(\tau+2r+1)-b\arg(-\tau+2r+1)\\ +(a+b)\arg(\tau-1)-(a+b)\arg(\tau+1), \end{aligned} \tag{2.25}$$

and the arguments $\arg(\,\cdot\,)$ take the value zero on $(1,2r+1)$. Formula (2.25) has the following geometrical interpretation in the notation of Fig. 2:

$$\operatorname{Im}f'(\tau)=b(\beta_-+\beta_+)+(a+b)(\alpha_+-\alpha_-)=b(\pi-\beta)+(a+b)\alpha, \tag{2.26}$$

whence $\operatorname{Im} f'(\tau) > 0$ if $\operatorname{Im}\tau > 0$ and $\operatorname{Im} f'(\tau) < 0$ if $\operatorname{Im}\tau < 0$. By (2.24) and (2.26), the set of solutions of equation (2.22) for every λ is symmetric with respect to the imaginary axis, and the symmetry $\tau \mapsto \bar{\tau}$ maps it onto the set of solutions of equation (2.22) with $-\lambda$ instead of λ. In what follows we deal with $\lambda > 0$ and the upper half-plane. The results thus obtained can b transferred to the case when $\lambda < 0$ by the symmetry with respect to the real axis.

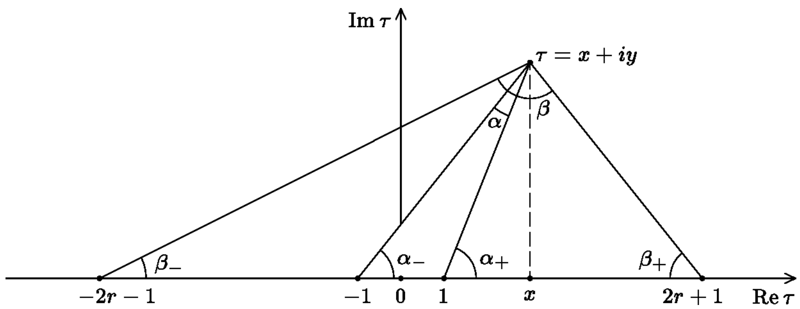

Figure 2

By (2.23), any solution of equation (2.22) is a root of the polynomial

$$(\tau + 2r + 1)^b(\tau - 1)^{a+b} - e^{\lambda\pi i}(-\tau + 2r + 1)^b(\tau + 1)^{a+b}, \tag{2.27}$$

whose degree is equal to $a + 2b - 1$ if λ is an integer that has even parity with b (otherwise, its degree is equal to $a + 2b$). If $\lambda \equiv b \pmod 2$, then this polynomial coincides with (2.21).

(a), (c). If $\lambda = 0$ (or, which is the same, $\lambda = a + b$), then the polynomial (2.27) is an odd function. Therefore, 0 is one of its roots. The corresponding solution $+i0$ of equation (2.22) corresponds to $\lambda = a + b$, since $\alpha = \beta = \pi$ in (2.26) for $\tau = 0$. Calculating the values of the polynomial at $1, 2r + 1$ and using the fact that this polynomial is odd, we obtain that it has a pair of real roots $\pm\mu_0$, where $\mu_0 \in (1, 2r + 1)$. We shall show later that μ_0 is the unique real root in $(1, 2r + 1)$, but until then we assume that μ_0 is the real root nearest to $2r + 1$ of the polynomial (2.27) with $\lambda = 0$ in the above interval. By (2.26), the solutions $-\mu_0 + i0$ and μ_0 correspond to $\lambda = 0$ in equation (2.22).

(b) Calculating the values of (2.27) with $\lambda = b$ at $2r + 1, +\infty$, we likewise obtain that this polynomial has a pair of real roots $\pm\mu_1$. We assume for the moment that among the roots in the interval $(2r + 1, +\infty)$ the root $\mu_1 \in (2r + 1, +\infty)$ is the nearest to $2r + 1$. By (2.26), the solutions $-\mu_1 + i0$ and $\mu_1 + i0$ correspond to $\lambda = b$ in equation (2.22).

In the case when $a \geqslant 3rb$ there is a better localization of the root μ_1, namely, $\mu_1 \in (2r + 1, 4r + 1)$. Indeed, the value of the polynomial (2.27) at $4r + 1$ is equal to

$$(6r+2)^b(4r)^{a+b} - (2r)^b(4r+2)^{a+b} = (2r)^b(4r)^{a+b}\biggl(\Bigl(3+\frac{1}{r}\Bigr)^b - \Bigl(1+\frac{1}{2r}\Bigr)^{a+b}\biggr) < 0,$$

since

$$(3r+1)\log\biggl(1+\frac{1}{2r}\biggr)=4\log\frac{3}{2}>2\log 2=\log\biggl(3+\frac{1}{r}\biggr) \qquad \text{if } r=1,$$
$$(3r+1)\log\biggl(1+\frac{1}{2r}\biggr)>\frac{3r+1}{2r+1}\geqslant\frac{7}{5}>2\log 2>\log\biggl(3+\frac{1}{r}\biggr) \qquad \text{if } r\geqslant 2,$$

where we have used the elementary inequality

$$\frac{1}{n+1}<\log\biggl(1+\frac{1}{n}\biggr)<\frac{1}{n}, \tag{2.28}$$

which holds for all positive integers n, whence

$$(a+b)\log\biggl(1+\frac{1}{2r}\biggr)\geqslant(3r+1)b\log\biggl(1+\frac{1}{2r}\biggr)>b\log\biggl(3+\frac{1}{r}\biggr).$$

(d) By (2.24), we have $\operatorname{Re} f'(\tau)=0$ on the imaginary axis. As τ ranges over the ray iy, $y>0$, the angles α,β decrease continuously from π to 0. Therefore, the function $\operatorname{Im} f'(\tau)=b(\pi-\beta)+(a+b)\alpha$ takes all intermediate values in the interval $(b\pi,(a+b)\pi)$. In particular, for every real λ, $b<\lambda<a+b$, we obtain at least one solution of equation (2.22) that lies on the imaginary axis.

(e) We shall now localize the complex solutions of (2.22) corresponding to the real λ, $0<|\lambda|<b$.

Consider the semicircle of radius 2ρ with centre $2r+1$, where $\rho<r$, in the upper half-plane. For $\tau=x+iy$ on this semicircle we have

$$\begin{gathered} |-\tau+2r+1|^2=4\rho^2, \qquad |\tau+2r+1|^2=4\rho^2+4(2r+1)x, \\ |\tau-1|^2=4\rho^2+4rx-4r(r+1), \qquad |\tau+1|^2=4\rho^2+4(r+1)x-4r(r+1). \end{gathered} \tag{2.29}$$

By (2.24),

$$2\operatorname{Re} f'(\tau)=\log\frac{(\rho^2+(2r+1)x)^b(\rho^2+rx-r(r+1))^{a+b}}{\rho^{2b}(\rho^2+(r+1)x-r(r+1))^{a+b}}. \tag{2.30}$$

We denote the function in (2.30) by $\tilde f(x)$, $2r+1-2\rho<x<2r+1+2\rho$. It is a monotonically increasing function of x, since

$$\begin{aligned} \tilde f'(x)&=\frac{b(2r+1)}{\rho^2+(2r+1)x}+\frac{(a+b)r}{\rho^2+rx-r(r+1)}-\frac{(a+b)(r+1)}{\rho^2+(r+1)x-r(r+1)} \\ &=\frac{b(2r+1)}{\rho^2+(2r+1)x}+\frac{(a+b)(r(r+1)-\rho^2)}{(\rho^2+rx-r(r+1))(\rho^2+(r+1)x-r(r+1))} \\ &=\frac{4b(2r+1)}{|\tau+2r+1|^2}+\frac{16(a+b)(r(r+1)-\rho^2)}{|\tau-1|^2|\tau+1|^2}>0. \end{aligned}$$

Therefore,

$$\tilde{f}(2r+1-2\rho) < \tilde{f}(x) < \tilde{f}(2r+1+2\rho), \qquad x \in (2r+1-2\rho, 2r+1+2\rho), \tag{2.31}$$

and

$$\operatorname{Re} f'(2r+1-2\rho) < \operatorname{Re} f'(\tau) < \operatorname{Re} f'(2r+1+2\rho+i0) \tag{2.32}$$

on the semicircle under consideration. Consider $\rho = (\mu_1 - 2r - 1)/2$, where $\mu_1 \in (2r+1, +\infty)$ is the real root of the polynomial (2.27) with $\lambda = b$. As mentioned above, $2r+1 < \mu_1 < 4r+1$ if $a \geqslant 3rb$, that is, the assumption $\rho < r$ holds. By (2.32), we have

$$\operatorname{Re} f'(\tau) < \operatorname{Re} f'(\mu_1 + i0) = 0 \tag{2.33}$$

for the points of the semicircle. Besides,

$$\lim_{\tau \to 2r+1} \operatorname{Re} f'(\tau) = +\infty. \tag{2.34}$$

Consider the rays starting from the point $2r+1$ and contained in the upper half-plane. By (2.33) and (2.34), each of these contains a point τ belonging to the domain bounded by the above semicircle and the real axis and such that $\operatorname{Re} f'(\tau) = 0$. Hence, this domain contains a part of the continuously differentiable curve

$$\operatorname{Re} f'(\tau) = \log \frac{|\tau + 2r + 1|^b |\tau - 1|^{a+b}}{|-\tau + 2r + 1|^b |\tau + 1|^{a+b}} = 0. \tag{2.35}$$

(See Fig. 3, where this part of the curve is situated, according to (2.31), between the two semicircles corresponding to $\rho = (\mu_1 - 2r - 1)/2$ and $\rho = (2r + 1 - \mu_0)/2$.) By (2.26), the values of $\operatorname{Im} f'(\tau)$ on this curve as it approaches the real axis are equal to 0 (at μ_0) and $b\pi$ (at μ_1). Therefore, $\operatorname{Im} f'(\tau)$ takes on the curve all intermediate values between 0 and $b\pi$, that is, the values $\lambda\pi$, where $0 < \lambda < b$. So we have produced solutions of equation (2.22) for these λ. Every point obtained from these by reflection in the imaginary axis also is a solution.

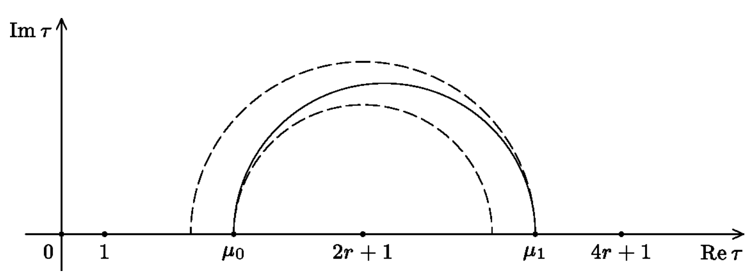

Figure 3

We have produced all the solutions of equation (2.22) listed in (a)–(e). To complete the proof, we have to show that there are no others. Assume the contrary, that is, assume that for some $\lambda_0 \in \mathbb{R}$ there is a solution of equation (2.22) that is not listed in (a)–(e). As mentioned above, this solution is a root of the polynomial (2.27) with $\lambda = \lambda_0$. We claim that the number of roots of (2.27) listed in (a)–(e) is equal to its degree, which will contradict our assumption.

If λ_0 is not an integer, then we have a purely imaginary solutions (see (d)) corresponding to $\lambda \equiv \lambda_0 \pmod 2$, $b < |\lambda| < a + b$, and b pairs of complex solutions (see (e)) corresponding to $\lambda \equiv \lambda_0 \pmod 2$, $|\lambda| < b$. Therefore, the total number of roots of (2.27) with $\lambda = \lambda_0$ listed in (d) and (e) is equal to the degree $a + 2b$ of the polynomial.

If λ_0 is an integer (even or odd), then a solution of equation (2.22) with $\lambda = \lambda_0$ is a root of the polynomial

$$(\tau + 2r + 1)^{2b}(\tau - 1)^{2(a+b)} - (\tau - 2r - 1)^{2b}(\tau + 1)^{2(a+b)}. \tag{2.36}$$

For this polynomial we have found two real roots $\pm\mu_0$ in (a), two real roots $\pm\mu_1$ in (b), one real root 0 in (c), $2(a-1)$ purely imaginary roots in (d) and $4(b-1)$ complex roots in (e). The total number of roots of the polynomial (2.36) listed in (a)–(e) is equal to its degree $2a + 4b - 1$.

Hence, any solution of equation (2.22) is contained in the list (a)–(e). This contradiction completes the proof of the lemma.

Remark 2.2. We replaced the original assumption $a \geqslant 2rb$ by the stronger assumption $a \geqslant 3rb$ in Lemma 2.7 only in order to localize the root μ_1. Therefore, Lemma 2.7 remains valid in the case when $a \geqslant 2rb$ if

$$\mu_1 < 4r + 1. \tag{2.37}$$

Corollary 2.2. *The set of $\tau \in \mathbb{C}$ for which* (2.35) *holds is the union of the imaginary axis and a pair of closed curves symmetric with respect to the imaginary axis* (*see Fig.* 4). *These curves divide the complex plane into four parts*, *in each of which the sign of* $\operatorname{Re} f'(\tau)$ *is constant.*

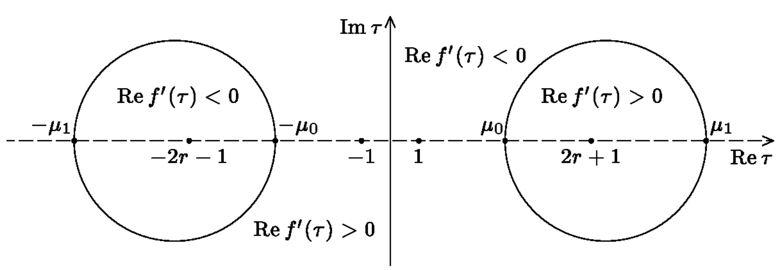

Figure 4

Corollary 2.3. *Any semicircle of radius* ρ, $2r+1-\mu_0<2\rho<\mu_1-2r-1$, *with centre* $2r+1$ *lying in the upper half-plane, intersects the curve* (2.35) (*see Fig.* 3) *at precisely one point.*

Proof. This assertion follows from inequalities (2.32), since the function $\operatorname{Re} f'(\tau)$ is monotonic on any of these semicircles.

Lemma 2.8. *Let* a,b,r,n *be positive integers,* $a\geqslant 2rb$, *let* $f(\tau),g(\tau)$ *be as defined in* (2.12), (2.13), *and let* $\mu,\lambda\in\mathbb{R}$, $1<\mu<2r+1$, $|\lambda|\leqslant b$. *Then the contour of integration* $\operatorname{Re}\tau=\mu$ *in the integral*

$$\int_{\mu-i\infty}^{\mu+i\infty} e^{n(f(\tau)-\lambda\pi i\tau)}g(\tau)\,\mathrm{d}\tau \tag{2.38}$$

can be replaced by any other contour $\mathcal{L}$ *joining infinitely remote points in the domains* $\operatorname{Re}\tau\geqslant\mu$, $\operatorname{Im}\tau<0$ *and* $\operatorname{Re}\tau\geqslant\mu$, $\operatorname{Im}\tau>0$ *and intersecting the real axis at precisely one point* $\tau=\mu$. *In particular,* $\mathcal{L}$ *can go along the upper and* (*or*) *lower bank of the cut* $[2r+1,+\infty)$.

Proof. For any real $N>\mu$ we join the points of the original contour $\operatorname{Re}\tau=\mu$ and those of the new contour $\mathcal{L}$ by two arcs of radius N with centre at the origin. We denote the corresponding points of intersection with the contours by $M_\pm$ and $L_\pm$ (see Fig. 5).

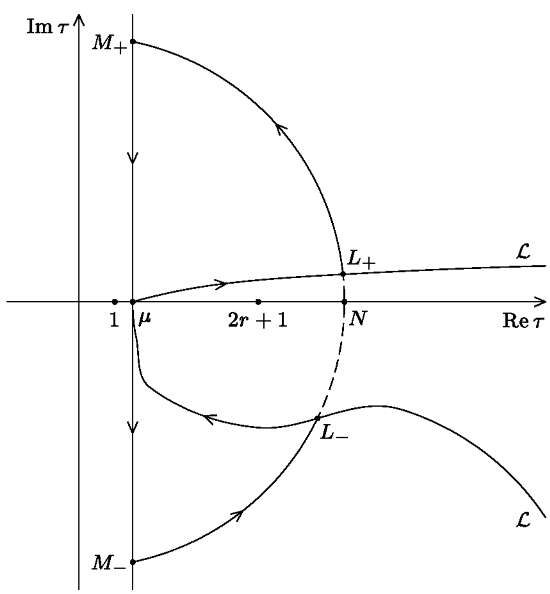

Figure 5

There are no singular points of the integrand in (2.38) inside the curvilinear triangles μL_+M_+ and μM_-L_-. The lemma will be proved if we can show that

$$\int_{L_+M_+} e^{n(f(\tau)-\lambda\pi i\tau)}g(\tau)\,\mathrm{d}\tau\to 0,\qquad \int_{M_-L_-} e^{n(f(\tau)-\lambda\pi i\tau)}g(\tau)\,\mathrm{d}\tau\to 0 \tag{2.39}$$

as $N\to\infty$, where L_+M_+ and M_-L_- are arcs of the circle of radius N with centre at the origin.

On the arcs L_+M_+ and M_-L_- of the circle $\tau=Ne^{it}$ we have the inequalities $0\leqslant t<\pi/2$ and $-\pi/2<t\leqslant 0$, respectively (the value $t=0$ is taken on the upper and lower bank of the cut $[2r+1,+\infty)$). Using Taylor's formula, we obtain the relations

$$\begin{aligned}
\log(\tau+1)&=\log(Ne^{it}+1)=\log(Ne^{it})+\log\biggl(1+\frac{e^{-it}}{N}\biggr)\\
&=\log N+it+\frac{e^{-it}}{N}+O\biggl(\frac{1}{N^2}\biggr), &(2.40)\\
\log(\tau-1)&=\log N+it-\frac{e^{-it}}{N}+O\biggl(\frac{1}{N^2}\biggr), &(2.41)\\
\log(\tau+2r+1)&=\log(Ne^{it}+2r+1)=\log(Ne^{it})+\log\biggl(1+\frac{(2r+1)e^{-it}}{N}\biggr)\\
&=\log N+it+\frac{(2r+1)e^{-it}}{N}+O\biggl(\frac{1}{N^2}\biggr), &(2.42)\\
\log(-\tau+2r+1)&=\mp\pi i+\log(Ne^{it}-(2r+1))\\
&=\mp\pi i+\log N+it-\frac{(2r+1)e^{-it}}{N}+O\biggl(\frac{1}{N^2}\biggr), &(2.43)
\end{aligned}$$

where $-$ $(+)$ in $\mp\pi i$ in (2.43) corresponds to the arc L_+M_+ (M_-L_-) according to the choice of branches of the logarithm. The constant in $O(1/N^2)$ in formulae (2.40)–(2.43) is absolute, since $|e^{-it}|=1$ for real t. Substituting these expansions into (2.12), we obtain the formula

$$\begin{aligned}
f(\tau)&=f(Ne^{it})\\
&=(\log N+it)\bigl(b(\tau+2r+1)+b(-\tau+2r+1)+(a+b)(\tau-1)-(a+b)(\tau+1)\bigr)\\
&\quad+\frac{e^{-it}}{N}\bigl(b(2r+1)(\tau+2r+1)-b(2r+1)(-\tau+2r+1)\\
&\qquad-(a+b)(\tau-1)-(a+b)(\tau+1)\bigr)\\
&\quad\mp\pi i\cdot b(-\tau+2r+1)+(a-2rb)2\log 2+O\biggl(\frac{|\tau|}{N^2}\biggr)\\
&=-2(a-2rb)\biggl(\log\frac{Ne}{2}+it\biggr)\pm b\pi i(Ne^{it}-2r-1)+O\biggl(\frac{1}{N}\biggr),
\end{aligned}$$

where the sign of $\pm b\pi$ coincides with the sign of t (and $\sin t$). Therefore,

$$\operatorname{Re}(f(\tau)-\lambda\pi i\tau) = -2(a-2rb)\log\frac{Ne}{2} - (b\mp\lambda)\pi N|\sin t| + O\biggl(\frac{1}{N}\biggr) \leqslant \frac{1}{n}$$

on the arcs L_+M_+ and M_-L_- as $N\to\infty$. (We have used the inequalities $a\geqslant 2rb$ and $|\lambda|\leqslant b$.) Hence, the estimate

$$|e^{n(f(\tau)-\lambda\pi i\tau)}| = e^{n\operatorname{Re}(f(\tau)-\lambda\pi i\tau)} \leqslant e \tag{2.44}$$

holds for all sufficiently large N. The following trivial estimate holds for the function (2.13) on the arcs L_+M_+ and M_-L_-:

$$|g(\tau)| = O\biggl(\frac{N^{b/2}\,N^{b/2}}{N^{(a+b)/2}\,N^{(a+b)/2}}\biggr) = O\biggl(\frac{1}{N^a}\biggr) = O\biggl(\frac{1}{N^2}\biggr) \qquad \text{as } N\to\infty, \tag{2.45}$$

since $a\geqslant 2rb\geqslant 2$. Since the length of each of the arcs L_+M_+ and M_-L_- does not exceed $\pi N/2$, estimates (2.44) and (2.45) imply that every integral in (2.39) is of order $O(1/N)$. Hence, the limiting relations (2.39) hold, which completes the proof of the lemma.

Lemma 2.9. *Let a,b,r be positive integers, $a\geqslant 2rb$, let $f(\tau),g(\tau)$ be the functions defined in* (2.12) *and* (2.13), *and let $\lambda\in\mathbb{R}$, $|\lambda|\leqslant b$. Assume, moreover, that* (2.37) *and the assumption*

$$\mu_0+\sqrt{\mu_0^2-1}\geqslant\mu_1 \tag{2.46}$$

hold for the real roots $\mu_0\in(1,2r+1)$ and $\mu_1\in(2r+1,+\infty)$ of (2.36).

Then the asymptotic behaviour of the integral (2.38) *with $\mu=\mu_0$ as $n\to\infty$ is determined by the single saddle point τ_0 (the solution of equation* (2.22)*) in the domain* $\operatorname{Im}\tau>0$. *More precisely, the following asymptotic formula holds*:

$$\begin{aligned} J_{n,\lambda} &= \frac{1}{2\pi i}\int_{\mu_0-i\infty}^{\mu_0+i\infty} e^{n(f(\tau)-\lambda\pi i\tau)}g(\tau)\,d\tau \\ &= (2\pi)^{-1/2}|f''(\tau_0)|^{-1/2}e^{n\operatorname{Re}f_0(\tau_0)}|g(\tau_0)|\cdot e^{-\frac{i}{2}\arg f''(\tau_0)+i\arg g(\tau_0)+in\operatorname{Im}f_0(\tau_0)} \\ &\qquad\times n^{-1/2}\bigl(1+O(n^{-1})\bigr) \qquad \text{as } n\to\infty, \end{aligned} \tag{2.47}$$

where

$$\begin{aligned} f_0(\tau) &= f(\tau)-f'(\tau)\tau \\ &= b(2r+1)\log(\tau+2r+1)+b(2r+1)\log(-\tau+2r+1) \\ &\qquad -(a+b)\log(\tau+1)-(a+b)\log(\tau-1)+(a-2rb)2\log 2. \end{aligned} \tag{2.48}$$

Proof. We shall prove the lemma only for $\lambda\geqslant 0$. The case when $\lambda<0$ can be reduced to this case by reflection in the real axis.

According to Lemma 2.8, we replace the contour of integration in (2.38) with $\mu = \mu_0$ by a contour consisting of rectilinear rays and segments. Let us note at once that the derivative of $\operatorname{Re} f(\tau)$ on every such rectilinear part $\tau = \tau_1 + e^{i\varphi}t$, $t \in \mathbb{R}$, is given by the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re} f(\tau) = \operatorname{Re}\frac{\mathrm{d}f(\tau)}{\mathrm{d}t} = \operatorname{Re}\biggl(\frac{\mathrm{d}\tau}{\mathrm{d}t}\cdot\frac{\mathrm{d}f(\tau)}{\mathrm{d}\tau}\biggr) = \operatorname{Re} f'(\tau)\,\cos\varphi - \operatorname{Im} f'(\tau)\,\sin\varphi. \tag{2.49}$$

If $\lambda = 0$, then μ_0 is the unique maximum point of the function $\operatorname{Re} f(\tau)$ on the line $\tau = \mu_0 + it$, $-\infty < t < +\infty$, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re} f(\tau) = -\operatorname{Im} f'(\tau)$$

and $\operatorname{Im} f'(\tau)$ takes negative values in the domain $\operatorname{Im}\tau < 0$ and positive ones in the domain $\operatorname{Im}\tau > 0$. So, in the case when $\lambda = 0$, we use the original contour $\operatorname{Re}\tau = \mu_0$.

If $\lambda = b$, then we replace the ray going from μ_0 to $\mu_0 + i\infty$ in the contour of integration in (2.38) with $\mu = \mu_0$ by a ray going from μ_0 to $+\infty$ along the upper bank of the cut $[2r+1, +\infty)$. The function

$$\operatorname{Re}(f(\tau) - \lambda\pi i\tau) \tag{2.50}$$

increases on the set $\tau = \mu_0 + it$, $-\infty < t \leqslant 0$, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}(f(\tau) - \lambda\pi i\tau) = -\operatorname{Im} f'(\tau) + \lambda\pi \geqslant \lambda\pi > 0. \tag{2.51}$$

On the set $\tau = t$, $\mu_0 \leqslant t < +\infty$, the function $\operatorname{Re}(f(\tau) - \lambda\pi i\tau) = \operatorname{Re} f(\tau)$ has a unique maximum point at $t = \mu_1$ (by Lemma 2.7 and Corollary 2.2).

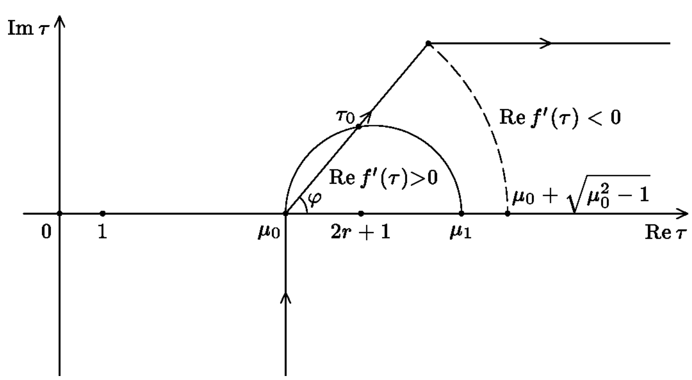

Figure 6

It remains to consider the case when $\lambda \in (0, b)$. We replace the ray going from μ_0 to $\mu_0 + i\infty$ by the segment $\tau = \mu_0 + e^{i\varphi}t$, $0 \leqslant t \leqslant \sqrt{\mu_0^2 - 1}$, passing through the saddle point τ_0 (the unique solution of equation (2.22) in the domain

$\operatorname{Re}\tau > 0$) and the ray $\tau = e^{i\varphi}\sqrt{\mu_0^2 - 1} + t$, $\mu_0 \leqslant t < +\infty$. By Corollary 2.3, the ray $\tau = \mu_0 + e^{i\varphi}t$, $t > 0$, intersects the curve $\operatorname{Re} f'(\tau) = 0$ at a single point (namely, τ_0), and $0 < \varphi < \pi/2$. By (2.46), τ_0 is an interior point of the above segment (see Fig. 6). On the ray $\tau = \mu_0 + it$, $t \leqslant 0$, the function (2.50) increases by (2.51). On the ray $\tau = e^{i\varphi}\sqrt{\mu_0^2 - 1} + t$, $t \geqslant \mu_0$, it decreases, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}(f(\tau) - \lambda\pi i\tau) = \operatorname{Re}(f'(\tau) - \lambda\pi i) = \operatorname{Re} f'(\tau) < 0$$

by Corollary 2.2. Hence, it is sufficient to show that τ_0 is the unique maximum point of the function (2.50) on the segment $\tau = \mu_0 + e^{i\varphi}t$, $0 \leqslant t \leqslant t_1$, where $t_1 = \sqrt{\mu_0^2 - 1}$. Assume that t_0 corresponds to the point $\tau_0 = \mu_0 + e^{i\varphi}t_0$ on the segment under consideration. By Corollary 2.3,

$$\operatorname{Re} f'(\mu_0 + e^{i\varphi}t) < 0 \quad \text{for } 0 < t < t_0, \qquad \operatorname{Re} f'(\mu_0 + e^{i\varphi}t) > 0 \quad \text{for } t_0 < t \leqslant t_1. \tag{2.52}$$

We claim that the function $\operatorname{Im} f'(\tau)$ increases monotonically on the segment under consideration. Let us use the geometric interpretation (2.26) of the function $\operatorname{Im} f'(\tau)$ once again. As $t \geqslant 0$ increases on the ray $\tau = \mu_0 + e^{i\varphi}t$, the angle β decreases monotonically from π to 0, whereas the angle α first increases from 0 to some α_0 and then decreases from α_0 to 0. We claim that the maximal value α_0 is attained on the ray at $t = t_1$, which will imply that the function

$$\operatorname{Im} f'(\tau) = b(\pi - \beta) + (a + b)\alpha$$

is monotonically increasing on the segment $\tau = \mu_0 + e^{i\varphi}t$, $0 \leqslant t \leqslant t_1$.

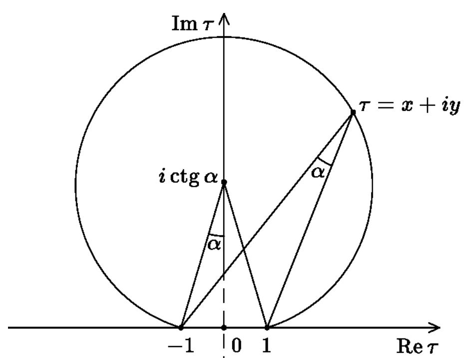

Figure 7

The points $\tau = x + iy$ of the upper half-plane at which the real segment $[-1, 1]$ subtends less than a given angle α belong to an arc of the circle of radius $1/\sin\alpha$ with centre $i\cot\alpha$ (this assertion for an acute angle α is depicted in Fig. 7):

$$x^2 + y^2 - 2y\cot\alpha = 1,$$

whence

$$\alpha = \cot^{-1}\frac{x^2+y^2-1}{2y}.$$

On the given ray, $x=\mu_0+t\cos\varphi$ and $y=t\sin\varphi$ if $t\geqslant 0$, whence

$$\begin{aligned}\frac{\mathrm{d}\alpha}{\mathrm{d}t} &= -\frac{(2x\cos\varphi+2y\sin\varphi)\cdot 2y - 2\sin\varphi\cdot(x^2+y^2-1)}{(x^2+y^2-1)^2+(2y)^2}\\ &= -2\sin\varphi\cdot\frac{t^2-(\mu^2-1)}{(x^2+y^2-1)^2+4y^2}.\end{aligned}$$

Therefore, the maximal value of α is attained at $t=\sqrt{\mu_0^2-1}$.

Since $\operatorname{Im} f'(\tau)$ is monotonically increasing on the segment $\tau=\mu_0+e^{i\varphi}t$, $0\leqslant t\leqslant t_1$, we have

$$\operatorname{Im} f'(\mu_0+e^{i\varphi}t)<\lambda\pi \quad \text{for } 0\leqslant t<t_0, \qquad \operatorname{Im} f'(\mu_0+e^{i\varphi}t)>\lambda\pi \quad \text{for } t_0<t\leqslant t_1. \tag{2.53}$$

Combining (2.52), (2.53) and (2.49), we obtain that the function

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Re}(f(\tau)-\lambda\pi i\tau)=\operatorname{Re} f'(\tau)\,\cos\varphi-(\operatorname{Im} f'(\tau)-\lambda\pi)\,\sin\varphi$$

on the segment $\tau=\mu_0+e^{i\varphi}t$, $0\leqslant t\leqslant t_1$, changes sign from plus to minus only when τ passes through $\tau_0=\mu_0+e^{i\varphi}t_0$. Hence, τ_0 is indeed the unique maximum point on the segment and on the whole contour of integration consisting of this segment and two rays.

To prove the asymptotic formula (2.47), we write down the contribution of the saddle point τ_0:

$$(2\pi)^{1/2}e^{\frac{\pi i}{2}-\frac{i}{2}\arg f''(\tau_0)}|f''(\tau_0)|^{-1/2}e^{n(f(\tau_0)-\lambda\pi i\tau_0)}g(\tau_0)\,n^{-1/2}\bigl(1+O(n^{-1})\bigr) \tag{2.54}$$

(see, for example, [23], § 5.7, formula (5.7.2)). We have

$$\begin{aligned}f''(\tau) &= \frac{b}{\tau+2r+1}+\frac{b}{-\tau+2r+1}+\frac{a+b}{\tau-1}-\frac{a+b}{\tau+1}\\ &= 2\frac{(a-2rb)\tau^2-(2r+1)(a-2rb+2ra)}{(\tau^2-1)(\tau^2-(2r+1)^2)}\neq 0\end{aligned}$$

at those τ for which $\operatorname{Re} f'(\tau)=0$ (in particular, τ_0). Using the equality

$$f(\tau_0)-\lambda\pi i\tau_0=f(\tau_0)-f'(\tau_0)\tau_0=f_0(\tau_0)$$

and separating the modulus and argument in (2.54), we obtain (2.47), which completes the proof of the lemma.

Corollary 2.4. *Let the assumptions of Lemma* 2.9 *hold. Assume, moreover, that*

$$-\frac{1}{2}\arg f''(\tau_0)+\arg g(\tau_0)\not\equiv\frac{\pi}{2}\ (\operatorname{mod}\pi\mathbb{Z}) \quad \textit{or} \quad \operatorname{Im} f_0(\tau_0)\not\equiv 0\ (\operatorname{mod}\pi\mathbb{Z}). \tag{2.55}$$

Then the following limiting relation holds for the integral

$$\operatorname{Re} J_{n,\lambda} = \frac{1}{2}(J_{n,\lambda} + J_{n,-\lambda}) = \frac{1}{2\pi i}\int_{\mu_0 - i\infty}^{\mu_0 + i\infty} e^{nf(\tau)} \cos(\lambda\pi n\tau) g(\tau)\,\mathrm{d}\tau :$$

$$\begin{aligned}\varlimsup_{n\to\infty} \frac{\log|\operatorname{Re} J_{n,\lambda}|}{n} &= \operatorname{Re} f_0(\tau_0)\\ &= \log\frac{2^{2(a-2rb)}|\tau_0 + 2r + 1|^{b(2r+1)}| - \tau_0 + 2r + 1|^{b(2r+1)}}{|\tau_0+1|^{a+b}|\tau_0 - 1|^{a+b}}.\end{aligned} \tag{2.56}$$

Moreover, in the case when

$$-\frac{1}{2}\arg f''(\tau_0) + \arg g(\tau_0) \equiv 0 \ (\operatorname{mod} \pi\mathbb{Z}) \quad \textit{and} \quad \operatorname{Im} f_0(\tau_0) \equiv 0 \ (\operatorname{mod} \pi\mathbb{Z}), \tag{2.57}$$

the integral $J_{n,\lambda}$ has the real asymptotics (2.47), *which implies that the upper limit in* (2.56) *becomes a limit.*

Proof. Assumption (2.55) implies that the real part of the coefficient of $n^{-1/2}$ in the asymptotic formula (2.47) is non-zero for an infinite sequence of numbers n. It is on this sequence that the limiting relation (2.56) is attained. If (2.57) holds, then the principal term of the asymptotics (2.47) is a real number, and (2.56) with lim instead of $\varlimsup$ follows immediately from (2.47).

We now state our definitive results concerning

$$\varkappa = \varlimsup_{n\to\infty} \frac{\log|I_n|}{n} = \varlimsup_{n\to\infty} \frac{\log|\tilde{I}_n|}{n} \tag{2.58}$$

(see Lemma 2.5). If $b = 1$ or $b = 2$, then the results take a simple form, and the upper limits in (2.58) can be replaced by limits. Our assertions in these cases will be stated as separate propositions.

Proposition 2.1. *Let $b = 1$, let r be a positive integer, let $a > 2r$ be an odd integer, and let $\mu_1 \in (2r+1, +\infty)$ be a real root of the polynomial* (2.21). *Then*

$$\varkappa = \lim_{n\to\infty} \frac{\log|I_n|}{n} = \log\frac{2^{2(a-2r)}(\mu_1 + 2r + 1)^{2r+1}(\mu_1 - 2r - 1)^{2r+1}}{(\mu_1+1)^{a+1}(\mu_1 - 1)^{a+1}}. \tag{2.59}$$

Proof. In the case when $b = 1$, $\tau = \mu_1$ is the unique maximum point of the function $\operatorname{Re} f(\tau)$ on the contour of integration for calculating the asymptotics of the integral $J_{n,1}$ in the proof of Lemma 2.9 (the ray $\tau = \mu_0 + it$, $t \leqslant 0$, and the ray $\tau = t$, $t \geqslant \mu_0$, that goes along the upper bank of the cut $[2r+1, +\infty)$). This implies, in particular, that $f''(\mu_1) < 0$, whence

$$-\frac{1}{2}\arg f''(\mu_1) \equiv \frac{\pi}{2} \ (\operatorname{mod} \pi\mathbb{Z}).$$

In the case when $b = 1$ we have for the function (2.13) that

$$\arg g(\mu_1) = -\frac{\pi}{2} \qquad \text{and} \qquad \operatorname{Im} f_0(\mu_1) = -(2r+1)\pi.$$

Using Corollary 2.4 with $\lambda = 1$ and $\tau_0 = \mu_1$, we obtain that (2.57) holds and

$$\lim_{n\to\infty} \frac{\log|\operatorname{Re} J_{n,1}|}{n} = \operatorname{Re} f_0(\mu_1).$$

The formula $\tilde{I}_n = -\operatorname{Re} J_{n,1}$ (see Corollary 2.1) and Lemma 2.5 imply that (2.59) holds.

Proposition 2.2. *Let $b = 2$, let r be a positive integer, assume that $a \geqslant 4r$ is an even integer, and let $\mu_0 \in (1, 2r+1)$ be a real root of the polynomial* (2.21). *Then*

$$\varkappa = \lim_{n\to\infty} \frac{\log|I_n|}{n} = \log \frac{2^{2(a-4r)}(\mu_0 + 2r + 1)^{2(2r+1)}(-\mu_0 + 2r + 1)^{2(2r+1)}}{(\mu_0 + 1)^{a+2}(\mu_0 - 1)^{a+2}}. \tag{2.60}$$

Proof. If $b = 2$, then $\tilde{I}_n = -J_{n,0}$, and $\tau = \mu_0$ is the unique maximum on the contour of integration $\tau = \mu_0 + it$, $t \in \mathbb{R}$, for calculating the asymptotics of the integral $J_{n,0}$ in the proof of Lemma 2.9. Therefore, $f''(\mu_0) > 0$, whence

$$-\frac{1}{2} \arg f''(\mu_0) \equiv 0 \ (\operatorname{mod} \pi\mathbb{Z}).$$

It is obvious that

$$\arg g(\mu_1) = 0, \qquad \operatorname{Im} f_0(\mu_1) = 0.$$

Corollary 2.4 with $\lambda = 0$ and $\tau_0 = \mu_0$ implies that (2.57) holds and

$$\lim_{n\to\infty} \frac{\log|J_{n,0}|}{n} = \operatorname{Re} f_0(\mu_0) = f_0(\mu_0).$$

Combining this with Lemma 2.5, we obtain (2.60).

Lemma 2.10. *Assume that the real root $\mu_1 \in (2r+1, +\infty)$ of the polynomial* (2.21) *satisfies the inequality*

$$\mu_1 \leqslant 2r + 1 + \min\left\{\frac{br(r+1)}{2(a+b)}, \frac{r(r+1)}{3(2r+1)}\right\}. \tag{2.61}$$

Then $\operatorname{Re} f_0(\tau)$ *regarded as a function of* $\operatorname{Re}\tau$ *increases in the domain* $\operatorname{Re}\tau > 0$, $\operatorname{Im}\tau \geqslant 0$ *on curve* (2.35).

Proof. As was shown in the proof of Lemma 2.7 (see also Corollary 2.3), on the smooth curve (2.35) in the domain $\operatorname{Re}\tau > 0, \operatorname{Im}\tau \geqslant 0$ the quantity $\rho = |\tau - (2r+1)|/2$, $\tau = x + iy$, can be represented as an implicit function of x, $\mu_0 \leqslant x \leqslant \mu_1$, which is continuously differentiable and increases. Using formulae (2.29), (2.30) and differentiating with respect to x, we obtain the following formula on the curve defined by formula (2.35):

$$2\rho\rho'\biggl(\frac{b}{\rho^2+(2r+1)x}-\frac{b}{\rho^2}+\frac{a+b}{\rho^2+rx-r(r+1)}-\frac{a+b}{\rho^2+(r+1)x-r(r+1)}\biggr)\\ +\biggl(\frac{b(2r+1)}{\rho^2+(2r+1)x}+\frac{(a+b)r}{\rho^2+rx-r(r+1)}-\frac{(a+b)(r+1)}{\rho^2+(r+1)x-r(r+1)}\biggr)=0.$$

Simple transformations yield that

$$2\rho\rho' x\biggl(\frac{b(2r+1)}{|\tau\pm(2r+1)|^2}-\frac{a+b}{|\tau\pm1|^2}\biggr)=\frac{b(2r+1)\rho^2}{|\tau\pm(2r+1)|^2}+\frac{(a+b)(r(r+1)-\rho^2)}{|\tau\pm1|^2}. \tag{2.62}$$

Therefore, the function $\operatorname{Re} f_0(\tau)$ can be regarded on the curve (2.35) as a function of x, $\mu_0\leqslant x\leqslant\mu_1$. By (2.29), we have

$$\tilde f_0(x):=2\operatorname{Re} f_0(\tau)=\log\frac{\rho^{2b(2r+1)}(\rho^2+(2r+1)x)^{b(2r+1)}}{(\rho^2+rx-r(r+1))^{a+b}(\rho^2+(r+1)x-r(r+1))^{a+b}}.$$

Combining this with (2.29) and (2.62), we obtain that

$$\begin{aligned}\tilde f_0'(x)&=2\rho\rho'\biggl(\frac{b(2r+1)}{\rho^2}+\frac{b(2r+1)}{\rho^2+(2r+1)x}\\&\qquad-\frac{a+b}{\rho^2+rx-r(r+1)}-\frac{a+b}{\rho^2+(r+1)x-r(r+1)}\biggr)\\&\qquad+\biggl(\frac{b(2r+1)^2}{\rho^2+(2r+1)x}-\frac{(a+b)r}{\rho^2+rx-r(r+1)}-\frac{(a+b)(r+1)}{\rho^2+(r+1)x-r(r+1)}\biggr)\\&=32\rho\rho'(2\rho^2+(2r+1)x)\biggl(\frac{b(2r+1)}{|\tau\pm(2r+1)|^2}-\frac{a+b}{|\tau\pm1|^2}\biggr)\\&\qquad+16(2r+1)\biggl(\frac{b(2r+1)\rho^2}{|\tau\pm(2r+1)|^2}+\frac{(a+b)(r(r+1)-\rho^2)}{|\tau\pm1|^2}\biggr)\\&\qquad+\frac{64\rho\rho'(a+b)r(r+1)}{|\tau\pm1|^2}-\frac{32(a+b)r(r+1)x}{|\tau\pm1|^2}\\&=\frac{32(\rho^2+(2r+1)x)}{x}\biggl(\frac{b(2r+1)\rho^2}{|\tau\pm(2r+1)|^2}+\frac{(a+b)(r(r+1)-\rho^2)}{|\tau\pm1|^2}\biggr)\\&\qquad+\frac{32(a+b)r(r+1)}{|\tau\pm1|^2}(2\rho\rho'-x).\end{aligned} \tag{2.63}$$

The function $\rho=\rho(x)$ increases on the curve (2.35). Hence, $\rho'\geqslant 0$. Continuing the chain in (2.63), we obtain that

$$\begin{aligned}\tilde f_0'(x) &\geqslant \frac{32(\rho^2+(2r+1)x)}{x}\,\frac{b(2r+1)\rho^2}{|\tau\pm(2r+1)|^2}+32(2r+1)\frac{(a+b)(r(r+1)-\rho^2)}{|\tau\pm 1|^2}\\ &\quad-\frac{32(a+b)r(r+1)x}{|\tau\pm 1|^2}\\ &=\frac{2b(2r+1)}{x}-\frac{2(a+b)((2r+1)\rho^2+r(r+1)x-r(r+1)(2r+1))}{(\rho^2+rx-r(r+1))(\rho^2+(r+1)x-r(r+1))}.\end{aligned}$$

Put

$$\varepsilon=\min\biggl\{\frac{br(r+1)}{2(a+b)},\frac{r(r+1)}{3(2r+1)}\biggr\}. \tag{2.64}$$

We have $\varepsilon\geqslant\mu_1-2r-1$. The localization of the curve (2.35) (see Corollary 2.3) implies that

$$2r+1-\varepsilon\leqslant x\leqslant 2r+1+\varepsilon,\qquad 0<\rho\leqslant\frac{\varepsilon}{2},$$

whence

$$\begin{aligned}\tilde f_0'(x) &> \frac{2b(2r+1)}{2r+1+\varepsilon}-\frac{2(a+b)((2r+1)\varepsilon/4+r(r+1))\varepsilon}{r(r+1)(r-\varepsilon)(r+1-\varepsilon)}\\ &>2(2r+1)\biggl(\frac{b}{2r+1+\varepsilon}-\frac{(a+b)(2r+1+\varepsilon)\varepsilon}{4r(r+1)(r-\varepsilon)(r+1-\varepsilon)}\biggr)\\ &=2(2r+1)\cdot\frac{4br(r+1)(r-\varepsilon)(r+1-\varepsilon)-\varepsilon(a+b)(2r+1+\varepsilon)^2}{4r(r+1)(r-\varepsilon)(r+1-\varepsilon)(2r+1+\varepsilon)}.\end{aligned} \tag{2.65}$$

Combining the inequalities

$$br(r+1)\geqslant\frac{\varepsilon}{2}(a+b)\qquad\text{if}\quad \varepsilon\leqslant\frac{br(r+1)}{2(a+b)},$$

$$8(r-\varepsilon)(r+1-\varepsilon)>(2r+1+\varepsilon)^2\qquad\text{if}\quad \varepsilon\leqslant\frac{r(r+1)}{3(2r+1)},$$

with (2.64) and (2.65), we obtain that $\tilde f_0'(x)>0$. Hence, $\operatorname{Re}f_0(\tau)$, regarded as a function of $x=\operatorname{Re}\tau$, increases on the curve (2.35), as was to be shown.

Proposition 2.3. *Let a,b,r be positive integers, assume that $a+b$ is even, $b\geqslant 3$, $a\geqslant 2rb$, and assume that* (2.61) *holds for the real root $\mu_1\in(2r+1,+\infty)$ of the polynomial* (2.21). *Let*

$$\varkappa:=\log\frac{2^{2(a-2rb)}|\tau_0+2r+1|^{b(2r+1)}|-\tau_0+2r+1|^{b(2r+1)}}{|\tau_0+1|^{a+b}|\tau_0-1|^{a+b}},$$

where τ_0 is a complex root of the polynomial (2.21) *in the domain $\operatorname{Re}\tau>0$, $\operatorname{Im}\tau>0$ with maximal real part $\operatorname{Re}\tau_0$. Assume, moreover, that* (2.55) *holds. Then*

$$\varlimsup_{n\to\infty}\frac{\log|I_n|}{n}=\varkappa. \tag{2.66}$$

Proof. The inequality (2.61) implies that $\mu_1 - (2r+1) < r/3$. Hence, (2.37) and (2.46) hold. By Lemma 2.10, the maximal value of the function $\operatorname{Re} f_0(\tau)$ on the roots of polynomial (2.21) is attained at $\tau = \tau_0$, for which $\lambda = k = b-2$. By Corollary 2.4, the asymptotics of the integral (2.20) is determined by the contribution of $\operatorname{Re} J_{n,b-2}$, since the contributions of the other quantities are exponentially smaller in comparison with $\operatorname{Re} J_{n,b-2}$, and the coefficient c_{b-2} in (2.20) is non-zero by assertion (b) of Lemma 2.2. Therefore,

$$\varlimsup_{n\to\infty} \frac{\log|\tilde{I}_n|}{n} = \varlimsup_{n\to\infty} \frac{\log|\operatorname{Re} J_{n,b-2}|}{n} = \varkappa.$$

Using Lemma 2.5, we obtain the desired relation (2.66).

§ 3. Estimates for the coefficients of linear forms

Our first result will be an upper estimate for the coefficients of linear forms.

Proposition 3.1. *Let a, b, r be positive integers, assume that $a+b$ is even, $a \geqslant 2rb$, and let the linear forms* (1.5) *be defined by* (1.2) *and* (1.1). *Then the following estimate holds for the coefficients $\bar{A}_s$, $s = 0$ or $s = b+1, \dots, a+b-1$:*

$$\varlimsup_{n\to\infty} \frac{\log|\bar{A}_s|}{n} \leqslant 2b(2r+1)\log(2r+1) + 2(a-2rb)\log 2, \qquad s = 0 \text{ or } s = b+1, \dots, a+b-1 \text{ and is odd.} \tag{3.1}$$

Moreover, estimate (3.1) *is exact in the case when a is even:*

$$\lim_{n\to\infty} \frac{\log|\bar{A}_s|}{n} = 2b(2r+1)\log(2r+1) + 2(a-2rb)\log 2, \qquad s = b+1, \dots, a+b-1 \text{ and is odd.} \tag{3.2}$$

Proof. Consider decomposition (1.6), where the coefficients can be calculated using the formulae

$$A_{k,j} = \frac{1}{(a-j)!} \frac{\mathrm{d}^{a-j}}{\mathrm{d}t^{a-j}} \bigl(R(t)(t+k)^a\bigr)\Big|_{t=-k}, \qquad k = 0, \pm 1, \dots, \pm n, \quad j = 1, 2, \dots, a.$$

First of all we claim that

$$\max_{k=0,\pm1,\dots,\pm n} |A_{k,a}| = |A_{0,a}| = \frac{((2r+1)n)!^{2b}(2n)!^{a-2rb}}{n!^{2(a+b)}}. \tag{3.3}$$

Since $|A_{k,a}| = |A_{-k,a}|$ for $k = 0, 1, \ldots, n$, it is sufficient to show that $|A_{k,a}|$ decreases as k increases from 0 to n. This can be verified directly:

$$\begin{aligned}\frac{|A_{k,a}|}{|A_{k-1,a}|} &= \frac{((2r+1)n+k)^b}{((2r+1)n-k+1)^b} \cdot \frac{(n-k+1)^{a+b}}{(n+k)^{a+b}} \\ &= \left(\frac{((2r+1)n+k)\cdot(n-k+1)}{((2r+1)n-k+1)\cdot(n+k)}\right)^b \cdot \left(\frac{n-k+1}{n+k}\right)^{a+b} \\ &= \left(\frac{(2r+1)n^2-k(k-1)+n(-2rk+2r+1)}{(2r+1)n^2-k(k-1)+n(2rk+1)}\right)^b \cdot \left(\frac{n-(k-1)}{n+k}\right)^{a+b} \\ &< 1, \qquad k = 1, \ldots, n.\end{aligned}$$

We now fix an integer k, $|k| \leqslant n$, and denote the logarithmic derivative of $R(t)(t+k)^a$ by $g_k(t)$:

$$g_k(t) = b \sum_{\substack{l=-(2r+1)n \\ l\ne k}}^{(2r+1)n} \frac{1}{t+l} - (a+b) \sum_{\substack{l=-n \\ l\ne k}}^{n} \frac{1}{t+l}.$$

Then for $j = 0, 1, 2, \ldots$ the modulus of the quantity

$$\begin{aligned}\frac{1}{j!}\,\frac{\mathrm{d}^j g_k(t)}{\mathrm{d}t^j}\bigg|_{t=-k} &= b \sum_{l=-(2r+1)n}^{-(n+1)} \frac{(-1)^j}{(l-k)^{j+1}} + b \sum_{l=n+1}^{(2r+1)n} \frac{(-1)^j}{(l-k)^{j+1}} \\ &\quad - a \sum_{\substack{l=-n \\ l\ne k}}^{n} \frac{(-1)^j}{(l-k)^{j+1}}\end{aligned}$$

is bounded above by $2(2rb+a)n$ (we estimate every summand on the right-hand side by 1). Using Leibniz' rule for the differentiation of a product, we obtain that

$$\begin{aligned}A_{k,a-j} &= \frac{1}{j!}\,\frac{\mathrm{d}^{j-1}}{\mathrm{d}t^{j-1}}\big(g_k(t)\cdot R(t)(t+k)^a\big)\bigg|_{t=-k} \\ &= \frac{1}{j} \sum_{m=0}^{j-1} \frac{1}{(j-1-m)!}\,\frac{\mathrm{d}^{j-1-m} g_k(t)}{\mathrm{d}t^{j-1-m}}\bigg|_{t=-k} \cdot A_{k,a-m}, \qquad j = 1, \ldots, a-1,\end{aligned} \tag{3.4}$$

whence

$$\begin{aligned}|A_{k,a-j}| &\leqslant 2(2rb+a)n \cdot \frac{1}{j} \sum_{m=0}^{j-1} |A_{k,a-m}| \\ &\leqslant 2(2rb+a)n \cdot \max_{m=0,1,\ldots,j-1} |A_{k,a-m}|, \qquad j = 1, \ldots, a-1.\end{aligned} \tag{3.5}$$

We obtain the following estimate from (3.5) by induction:

$$|A_{k,a-j}| \leqslant \big(2(2rb+a)n\big)^{j-1} |A_{k,a}|, \qquad j = 1, \ldots, a.$$

By (3.3), we have

$$|A_{k,a-j}| \leqslant \bigl(2(2rb+a)n\bigr)^{a-1} \frac{((2r+1)n)!^{2b}(2n)!^{a-2rb}}{n!^{2(a+b)}},$$
$$k = 0, \pm 1, \dots, \pm n, \quad j = 1, \dots, a.$$

Combining this with (1.11) and (1.12), we obtain that

$$|\bar{A}_s| \leqslant 2^{a+b-2}(2n+1)^2\bigl(2(2rb+a)n\bigr)^{a-1} \frac{((2r+1)n)!^{2b}(2n)!^{a-2rb}}{n!^{2(a+b)}}, \qquad s = 0 \text{ or } s = b+1, \dots, a+b-1. \tag{3.6}$$

We obtain the limiting relations (3.1) from (3.6) using Stirling's formula (2.17) for $\Gamma(z) = (z-1)!$ with positive integers $z \to \infty$.

If a is an even integer and j is an even integer such that $0 \leqslant j < a$, then formula (3.4) with $k = 0$ implies that $|A_{0,a-j}| \geqslant |A_{0,a}|$. If a is even and s is odd, $b < s < a+b$, then the summands on the right-hand side of (1.11) have the same sign, whence

$$|\bar{A}_s| \geqslant |A_{0,a}|, \qquad s = b+1, \dots, a+b-1 \text{ and is odd.}$$

So we have obtained both upper and lower estimates for the asymptotics of coefficients of linear forms (1.5) for even a using Stirling's formula. This proves the limiting relations (3.2) and completes the proof of the proposition.

It turns out that the estimate (3.1) holds for odd values of a as well. The results obtained in § 2 enable us to calculate the asymptotics of the coefficients (1.11) for odd a. This will occupy the rest of this section.

For positive integers m the following expansion holds in the neighbourhood of the point $t = 0$:

$$\left(\frac{\sin \pi t}{\pi}\right)^{2m} = \sum_{l=m}^{\infty} d_l^{(m)} t^{2l}, \tag{3.7}$$

where $d_l^{(m)}$, $l = m, m+1, \dots$, are real numbers, $d_m^{(m)} = 1$.

Lemma 3.1. *Assume that a is an odd integer. Then the following recurrence relations hold for the coefficients of the linear form* (1.5):

$$\bar{A}_{2m+b} = -\frac{(2m+b-1)!}{(2m)!\,(b-1)!} \frac{(-1)^b}{\pi i} \int_{iM-\infty}^{iM+\infty} \left(\frac{\sin \pi t}{\pi}\right)^{2m} R(t)\,\mathrm{d}t - \sum_{l=m+1}^{(a-1)/2} \frac{(2l)!\,(2m+b-1)!}{(2m)!\,(2l+b-1)!} d_l^{(m)} \bar{A}_{2l+b}, \qquad m = 1, 2, \dots, \frac{a-1}{2}, \tag{3.8}$$

where $M > 0$ is an arbitrary real constant.

Proof. If m is a positive integer, then (3.7) implies that the following expansion holds in the neighbourhood of $t = -k \in \mathbb{Z}$:

$$\left(\frac{\sin \pi t}{\pi}\right)^{2m} = \left(\frac{\sin \pi(t+k)}{\pi}\right)^{2m} = \sum_{l=m}^{\infty} d_l^{(m)} (t+k)^{2l}.$$

Formula (1.6) implies that the following formulae hold in the neighbourhood of $t=-k$ for the function defined by (1.1):

$$R(t)=\frac{A_{k,a}}{(t+k)^a}+\frac{A_{k,a-1}}{(t+k)^{a-1}}+\dots+\frac{A_{k,1}}{t+k}+O(1),$$

where $k=0,\pm1,\dots,\pm n$, and $R(t)=O(1)$ in the neighbourhood of $t=-k\in\mathbb{Z}$, $|k|>n$. Therefore,

$$\operatorname*{Res}_{t=-k}\biggl(\Bigl(\frac{\sin\pi t}{\pi}\Bigr)^{2m}R(t)\biggr)=\begin{cases}0 & \text{if } |k|>n,\\ \displaystyle\sum_{l=m}^{(a-1)/2}d_l^{(m)}A_{k,2l+1} & \text{if } |k|\leqslant n,\end{cases}$$

for $m<a/2$ and any integer k. If the closed contour $\mathcal{L}$ goes around the points $0,\pm1,\dots,\pm n$ anticlockwise, then

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\mathcal{L}}\Bigl(\frac{\sin\pi t}{\pi}\Bigr)^{2m}R(t)\,\mathrm{d}t&=(-1)^{b-1}\sum_{l=m}^{(a-1)/2}\frac{(2l)!\,(b-1)!}{(2l+b-1)!}d_l^{(m)}\bar{A}_{2l+b}\\&=(-1)^{b-1}\biggl(\frac{(2m)!\,(b-1)!}{(2m+b-1)!}\bar{A}_{2m+b}+\sum_{l=m+1}^{(a-1)/2}\frac{(2l)!\,(b-1)!}{(2l+b-1)!}d_l^{(m)}\bar{A}_{2l+b}\biggr),\end{aligned}\tag{3.9}$$

which follows from (1.11) and the relation $d_m^{(m)}=1$. We deduce from (3.9) the following recurrence formulae:

$$\begin{aligned}\bar{A}_{2m+b}&=\frac{(2m+b-1)!}{(2m)!\,(b-1)!}\,\frac{(-1)^b}{2\pi i}\oint_{\mathcal{L}}\Bigl(\frac{\sin\pi t}{\pi}\Bigr)^{2m}R(t)\,\mathrm{d}t\\&\quad-\sum_{l=m+1}^{(a-1)/2}\frac{(2l)!\,(2m+b-1)!}{(2m)!\,(2l+b-1)!}d_l^{(m)}\bar{A}_{2l+b},\qquad m=1,2,\dots,\frac{a-1}{2}.\end{aligned}\tag{3.10}$$

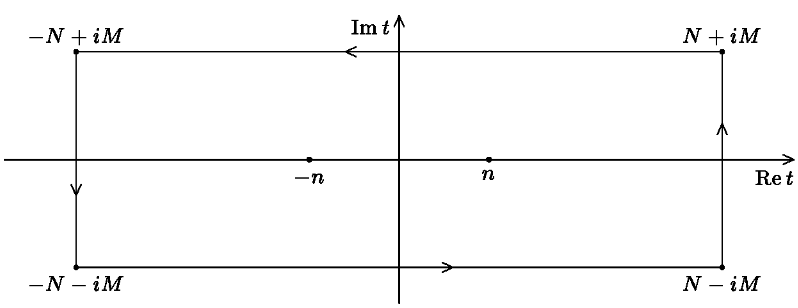

Figure 8

Now let the contour of integration $\mathcal{L}$ be the rectangle with vertices $\pm N \pm iM$, where $M > 0$ is a fixed real constant and $N > n$ is sufficiently large (see Fig. 8). As $N \to \infty$, the following estimates hold on the vertical sides of the rectangle:

$$\left|\frac{\sin \pi t}{\pi}\right| \leqslant \frac{e^{\pi M}}{\pi}, \qquad R(t) = O(N^{-2}).$$

Therefore, we have

$$\begin{aligned}\frac{1}{2\pi i} &\oint_{\mathcal{L}} \left(\frac{\sin \pi t}{\pi}\right)^{2m} R(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi i}\left(\int_{iM+N}^{iM-N} + \int_{-iM-N}^{-iM+N}\right)\left(\frac{\sin \pi t}{\pi}\right)^{2m} R(t)\,\mathrm{d}t + O(N^{-1}) \qquad \text{as } N \to \infty \end{aligned} \tag{3.11}$$

for $m = 1, 2, \dots, (a-1)/2$, where the constant in $O(N^{-1})$ depends only on M. Since the integrand in (3.11) is an odd function (see (1.8)) and the contour $\mathcal{L}$ is symmetric with respect to 0, it is sufficient to integrate over half the contour and double the result. Passing to the limit as $N \to \infty$, we obtain the recurrence formulae (3.8) from (3.10), which completes the proof of the lemma.

It follows from Lemma 3.1 that the asymptotic behaviour of the coefficients of the linear form (1.5) is closely connected with the asymptotics of the integrals

$$\begin{aligned} K_{n,m} &= \frac{1}{\pi i}\int_{iM-\infty}^{iM+\infty} \left(\frac{\sin \pi t}{\pi}\right)^{2m} R_n(t)\,\mathrm{d}t \\ &= \frac{(-1)^{bn}}{\pi i}\int_{iM-\infty}^{iM+\infty} \left(\frac{\sin \pi t}{\pi}\right)^{2m+b} \\ &\quad \times \frac{\Gamma(\pm t + (2r+1)n + 1)^b \Gamma(t-n)^{a+b} (2n)!^{a-2rb}}{\Gamma(t+n+1)^{a+b}}\,\mathrm{d}t, \quad m = 1, 2, \dots, \frac{a-1}{2}, \end{aligned} \tag{3.12}$$

as $n \to \infty$, where we have used formula (2.15). As in § 2, we assume that the cuts $(-\infty, n]$ and $[(2r+1)n, +\infty)$ are made in the t-plane.

Lemma 3.2 (compare [23], § 6.5). *For any $y_0 > 0$ the asymptotic equality*

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \log\sqrt{2\pi} + O\bigl(|z|^{-1}\bigr) + O\bigl(e^{-2\pi \operatorname{Im} z}\bigr) \tag{3.13}$$

holds in the domain $\operatorname{Im} z \geqslant y_0$.

Proof. The asymptotic relation (2.17) holds in the domain $|\arg z| < \pi - \varepsilon$, but the identity

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z \sin \pi z} = \frac{2\pi i}{z e^{-\pi i z}(1 - e^{2\pi i z})} \tag{3.14}$$

enables us to write down asymptotics in the quadrant $\operatorname{Im} z > 0$, $\operatorname{Re} z < 0$ as well. Making the cut $(-\infty, 0]$ in the z-plane and fixing principal values of the argument, we deduce from (2.17) and (3.14) the following relation in this quadrant:

$$\begin{aligned}\log\Gamma(z) &= \log(2\pi i) - \log z + \pi i z - \log(1-e^{2\pi i z}) - \log\Gamma(-z)\\ &= \biggl(\log(2\pi) + \pi i\Bigl(z+\frac{1}{2}\Bigr) - \log z - \log(1-e^{2\pi i z})\biggr)\\ &\qquad - \biggl(\Bigl(-z-\frac{1}{2}\Bigr)(\log z - \pi i) + z + \log\sqrt{2\pi} + O\bigl(|z|^{-1}\bigr)\biggr)\\ &= \Bigl(z-\frac{1}{2}\Bigr)\log z - z + \log\sqrt{2\pi} + O\bigl(|z|^{-1}\bigr) + \log(1-e^{2\pi i z}). \end{aligned} \tag{3.15}$$

Observing that $|e^{-2\pi i z}| \leqslant u_0 := e^{-2\pi y_0} < 1$ in the domain $\operatorname{Im} z \geqslant y_0$ and combining (3.15) with the relation $|\ln(1-u)| \leqslant C|u|$, which holds for $|u| \leqslant u_0$ with some constant C depending only on u_0, we complete the proof of formula (3.13).

Lemma 3.3. *Let μ be a positive real constant. Then the relation*

$$K_{n,m} = \tilde{K}_{n,m} \cdot \frac{(-1)^{bn}(2\sqrt{\pi n}\,)^{a-2rb}2^b}{\pi^{2m}n^{a-1}} \cdot \bigl(1 + O(n^{-1}) + O(e^{-2\pi n\mu})\bigr)$$

holds for the integrals (3.12) *as $n \to \infty$, where*

$$\tilde{K}_{n,m} = \frac{1}{\pi i}\int_{i\mu-\infty}^{i\mu+\infty} \sin^{2m+b}\pi n\tau \cdot e^{nf(\tau)}g(\tau)\,\mathrm{d}\tau, \qquad m = 1,2,\dots,\frac{a-1}{2}, \tag{3.16}$$

and the functions $f(\tau), g(\tau)$ are defined by (2.12) *and* (2.13) (*see also Remark* 2.1).

Proof. The proof repeats the proof of Lemma 2.5 except that we replace the asymptotics (2.17) by (3.13) (when integrating over the contour $\operatorname{Im} t = M = \mu n$, $\mu > 0$).

Lemma 3.4. *Assume that* (2.61) *holds for the real root $\mu_1 \in (2r+1, +\infty)$ of the polynomial* (2.21), *and assume that*

$$|\eta_1| < \min\biggl\{\sqrt{3}, \frac{\log(r^2+r)}{\pi}\biggr\} \tag{3.17}$$

for the purely imaginary root $\eta_1 \in (0, +i\infty)$ of minimal absolute value. Then the following relation holds for integrals (3.16) *as $n \to \infty$:*

$$\begin{aligned} |\tilde{K}_{n,(a-1)/2}| &= \frac{|g(\eta_1)|}{|f''(\eta_1)|^{1/2}2^{a+b-1/2}\pi^{1/2}n^{1/2}} e^{n\operatorname{Re} f_0(\eta_1)}\bigl(1+O(n^{-1})\bigr),\\ |\tilde{K}_{n,m}| &= |\tilde{K}_{n,(a-1)/2}| \cdot O(e^{-2\pi n\mu}), \qquad m = 1,2,\dots,\frac{a-1}{2}-1, \end{aligned} \tag{3.18}$$

where the constant μ is defined by the equality $i\mu = \eta_1$.

Proof. By Lemma 2.7, $\lambda = a+b-1$ in equation (2.22) corresponds to the root $i\mu = \eta_1$ of the polynomial (2.21).

Using the equality

$$\begin{aligned}\sin^{2m+b}\pi n\tau &= \left(\frac{e^{\pi in\tau}-e^{-\pi in\tau}}{2i}\right)^{2m+b}\\ &= \chi_m\frac{(-1)^{(a+b)/2}}{2^{a+b-1}i}e^{-\pi i(a+b-1)n\tau}+\sum_{l=1}^{a+b-1}h_{m,l}ie^{-\pi i(a+b-2l-1)n\tau},\\ &\qquad m=1,2,\dots,\frac{a-1}{2},\end{aligned}$$

where

$$\chi_m=\begin{cases}1 & \text{if } m=\dfrac{a-1}{2},\\ 0 & \text{otherwise}\end{cases}$$

and $h_{m,l}$, $m=1,\dots,(a-1)/2$, $l=1,\dots,a+b-1$ are real coefficients, we write the integrals (3.16) as

$$\begin{gathered}\tilde K_{n,m}=-\chi_m\frac{(-1)^{(a+b)/2}}{2^{a+b-1}\pi}J_{n,a+b-1}+\frac{1}{\pi}\sum_{l=1}^{a+b-1}h_{m,l}J_{n,a+b-2l-1},\\ m=1,2,\dots,\frac{a-1}{2},\end{gathered}\tag{3.19}$$

where

$$J_{n,k}=\int_{i\mu-\infty}^{i\mu+\infty}e^{n(f(\tau)-k\pi i\tau)}g(\tau)\,\mathrm{d}\tau,\qquad k=0,\pm1,\dots,\pm(a+b-1).\tag{3.20}$$

First we shall study the asymptotics of the integral $J_{n,a+b-1}$. The path of integration passes through the saddle point $\eta_1=i\mu$. We claim that $x=0$ is a maximum point of the function

$$\tilde f(x)=\operatorname{Re}(f(\tau)-(a+b-1)\pi i\tau)\big|_{\tau=x+i\mu}.\tag{3.21}$$

We have

$$\tilde f'(x)=\operatorname{Re}(f'(\tau)-(a+b-1)\pi i)\big|_{\tau=x+i\mu}=\operatorname{Re}f'(x+i\mu).$$

Hence, the sole candidates for maximum points of (3.21), besides $x=0$, are $x=\pm x_0$, where $x_0+i\mu$ is the point of intersection of the ray $\tau=x+i\mu,\ x>0$, with the curve $\operatorname{Re}f'(\tau)=0$ in the domain $\operatorname{Re}\tau>0$ (see Fig. 4) at which $\operatorname{Re}f'(\tau)$ changes the sign from + to − (if there is such a point). On the other hand,

$$f(\tau)-(a+b-1)\pi i\tau=f_0(\tau)+f'(\tau)\tau-(a+b-1)\pi i\tau$$

for the function $f_0(\tau)$ defined in (2.48). By Lemma 2.10 and the inequality $\operatorname{Im}f'(x_0+i\mu)>0$, we have

$$\begin{aligned}\tilde f(\pm x_0)=\tilde f(x_0)&=\operatorname{Re}f_0(x_0+i\mu)-\mu\operatorname{Im}f'(x_0+i\mu)+(a+b-1)\pi\mu\\ &<\operatorname{Re}f_0(\mu_1)+(a+b-1)\pi\mu.\end{aligned}\tag{3.22}$$

The inequality

$$2r+1<\mu_1\leqslant 2r+1+2\varepsilon, \qquad \varepsilon:=\frac{r(r+1)}{6(2r+1)}, \tag{3.23}$$

which follows from (2.61), enables us to obtain an upper estimate for $\operatorname{Re} f_0(\mu_1)$. We have

$$\operatorname{Re} f_0(\mu_1)<b(2r+1)\log((2r+1+\varepsilon)\varepsilon)-(a+b)\log((r+1)r)<-(a-2rb)\log(r^2+r),$$

since $(2r+1+\varepsilon)\varepsilon<r(r+1)$ by (3.23). Continuing estimate (3.22) and using inequality (3.17), we obtain the inequalities

$$\begin{aligned}\tilde f(\pm x_0)&<-(a-2rb)\log(r^2+r)+(a+b)\pi\mu\\&\leqslant -(a-2rb)\log(r^2+r)+(a+b)\log(r^2+r)\\&=b(2r+1)\log(r^2+r).\end{aligned} \tag{3.24}$$

To obtain a lower estimate for $\tilde f(0)$, we use the inequalities

$$|i\mu\pm(2r+1)|>2r+1, \qquad |i\mu\pm1|\leqslant|i\sqrt{3}\pm1|=2.$$

We have

$$\begin{aligned}\tilde f(0)=\operatorname{Re} f_0(i\mu)&>2(a-2rb)\log 2+2b(2r+1)\log(2r+1)-2(a+b)\log 2\\&=2b(2r+1)\log\left(r+\frac{1}{2}\right)>b(2r+1)\log(r^2+r).\end{aligned} \tag{3.25}$$

Comparing (3.24) with (3.25), we obtain that $\tilde f(0)>\tilde f(\pm x_0)$. Hence, the saddle point $\tau=i\mu$ is a maximum point of the function $\operatorname{Re}(f(\tau)-(a+b-1)\pi i\tau)$ on the contour $\operatorname{Im}\tau=\mu$. Therefore, $J_{n,a+b-1}$ is equal to the contribution of the saddle point $i\mu=\eta_1$:

$$\begin{aligned}J_{n,a+b-1}&=\frac{(2\pi)^{1/2}g(\eta_1)}{|f''(\eta_1)|^{1/2}n^{1/2}}e^{n\operatorname{Re}(f(\eta_1)-(a+b-1)\pi i\eta_1)}\bigl(1+O(n^{-1})\bigr)\\&=\frac{(2\pi)^{1/2}g(\eta_1)}{|f''(\eta_1)|^{1/2}n^{1/2}}e^{n\operatorname{Re} f_0(\eta_1)}\bigl(1+O(n^{-1})\bigr),\end{aligned} \tag{3.26}$$

and a similar estimate holds for the integral of the modulus of the integrand:

$$\int_{i\mu-\infty}^{i\mu+\infty}|e^{n(f(\tau)-(a+b-1)\pi i\tau)}g(\tau)|\,\mathrm{d}\tau=\frac{(2\pi)^{1/2}|g(\eta_1)|}{|f''(\eta_1)|^{1/2}n^{1/2}}e^{n\operatorname{Re} f_0(\eta_1)}\bigl(1+O(n^{-1})\bigr). \tag{3.27}$$

For the integrals (3.20) with $k<a+b-1$ we use the relation

$$|e^{n(f(\tau)-k\pi i\tau)}g(\tau)|=|e^{n(f(\tau)-(a+b-1)\pi i\tau)}g(\tau)|\cdot e^{-(a+b-1-k)n\mu}$$

on the contour $\operatorname{Im}\tau=\mu$. Combining this with (3.27) and (3.26), we obtain the inequality

$$|J_{n,k}|\leqslant Ce^{-(a+b-1-k)n\mu}|J_{n,a+b-1}|, \qquad k=0,\pm1,\dots,\pm(a+b-1), \tag{3.28}$$

with some constant $C>0$ that does not depend on n or k. (Inequality (3.28) is obtained by integration along a finite segment, say $[i\mu-1,i\mu+1]$, since the contribution of the integral over the remaining infinite part is exponentially small compared with the contribution of $i\mu$.) Substituting estimates (3.26) and (3.28) into (3.19), we obtain (3.18), which completes the proof of the lemma.

Proposition 3.2. *Let a, b, r be positive integers, let a, b be odd integers, $a > 2rb$, assume that the real root $\mu_1 \in (2r+1, +\infty)$ of the polynomial* (2.21) *is such that* (2.61) *holds, and assume that* (3.17) *holds for the purely imaginary root $\eta_1 \in (0, +i\infty)$ of minimal absolute value. Then the following asymptotic formula holds for the coefficients $\bar{A}_s$ of the linear form* (1.5):

$$\varlimsup_{n\to\infty} \frac{\log|\bar{A}_s|}{n} \leqslant \log \frac{2^{2(a-2rb)}|\eta_1+2r+1|^{b(2r+1)}|-\eta_1+2r+1|^{b(2r+1)}}{|\eta_1+1|^{a+b}|\eta_1-1|^{a+b}}, \qquad (3.29)$$

where $s = 0$ or $s = b+1, \dots, a+b-1$ and is odd. In the case when $s = a+b-1$ the upper limit becomes a limit and the inequality becomes an equality.

Proof. The limiting relation

$$\lim_{n\to\infty} \frac{\log|\bar{A}_{a+b-1}|}{n} = \operatorname{Re} f_0(\eta_1)$$

follows from formula (3.8) with $m = (a-1)/2$ by Lemmas 3.3 and 3.4. For the other odd integers s, $b < s < a+b-1$, estimates (3.29) follow from Lemmas 3.1, 3.3 and 3.4. In the case when $s = 0$ formula (1.5) implies that

$$|\bar{A}_0| \leqslant |I| + \sum_{\substack{s \text{ is odd} \\ b<s<a+b}} |\bar{A}_s|\zeta(s),$$

whence

$$\varlimsup_{n\to\infty} \frac{\log|\bar{A}_0|}{n} \leqslant \max\{\operatorname{Re} f_0(\tau_0), \operatorname{Re} f_0(\eta_1)\}, \qquad (3.30)$$

where the root τ_0 of polynomial (2.21) is defined in Proposition 2.3. Lemma 2.10 implies that $\operatorname{Re} f_0(\tau_0) < \operatorname{Re} f_0(\mu_1)$. The inequality $\operatorname{Re} f_0(\mu_1) < \operatorname{Re} f_0(\eta_1)$ (even a stronger one) was proved in Lemma 3.4. Therefore, the maximum on the right-hand side of (3.30) is equal to $\operatorname{Re} f_0(\eta_1)$, which completes the proof of the proposition.

Remark 3.1. Estimate (3.29) is but a slight improvement of (3.1), and will not be used in the proofs of Theorems 0.3 and 0.4. However, it is quite natural (and could have been foreseen) that the asymptotics of linear forms and their coefficients is determined by the values of the same function $\operatorname{Re} f_0(\tau)$ at different roots of the same polynomial (2.21). (In the case when a is an even integer the asymptotics of the coefficients of the linear form (1.5) is determined by the root $\eta_0 = 0$ of polynomial (2.21); see Proposition 3.1.)

§ 4. Refined estimates for the denominators of linear forms

The asymptotics of the denominators of linear forms (1.2) (as $n \to \infty$) obtained in Lemma 1.4, although somewhat coarse, is sufficient for the proof of Rivoal's theorem. We shall refine our results on denominators using the following generalization of Lemma 1.2.

Lemma 4.1. *Assume that for some polynomial $P(t)$, $\deg P(t) < m(n+1)$, the rational function*

$$R(t)=\frac{P(t)}{\bigl((t+s)(t+s+1)\cdots(t+s+n)\bigr)^m}$$

(*which may be reducible*) *satisfies the conditions*

$$\begin{gathered}\frac{D_n^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\bigl(R(t)(t+k)^m\bigr)\Big|_{t=-k}\in\mathbb{Z},\\ k=s,s+1,\dots,s+n,\quad j=0,1,\dots,m-1,\end{gathered}\tag{4.1}$$

where D_n is the least common multiple of the numbers $1,2,\dots,n$. Then

$$\frac{D_n^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\bigl(R(t)(t+k)^m\bigr)\Big|_{t=-k}\in\mathbb{Z},\qquad k=s,s+1,\dots,s+n,\tag{4.2}$$

for all non-negative integers j.

Proof. For the integers $j=0,1,\dots,m-1$, the inclusions (4.2) follow directly from (4.1). In what follows we consider only the integers $j\geqslant m$.

The rational function $R(t)$ can be decomposed into a sum of partial fractions as follows:

$$R(t)=\sum_{l=s}^{s+n}\biggl(\frac{B_{l,0}}{(t+l)^m}+\frac{B_{l,1}}{(t+l)^{m-1}}+\dots+\frac{B_{l,m-2}}{(t+l)^2}+\frac{B_{l,m-1}}{t+l}\biggr),\tag{4.3}$$

where

$$B_{k,j}=\frac{1}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\bigl(R(t)(t+k)^m\bigr)\Big|_{t=-k},\qquad k=s,s+1,\dots,s+n,\quad j=0,1,\dots,m-1.$$

Since differentiation is a linear operation and formulae (4.3) and (4.1) hold, it is sufficient to show that

$$\begin{gathered}\frac{D_n^{j-q}}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\biggl((t+k)^m\frac{1}{(t+l)^{m-q}}\biggr)\bigg|_{t=-k}\in\mathbb{Z},\\ q=0,1,\dots,m-1,\quad k,l=0,\pm1,\dots,\pm n.\end{gathered}\tag{4.4}$$

Since

$$\begin{aligned}(t+k)^m\frac{1}{(t+l)^{m-q}}&=\frac{((t+l)-(l-k))^m}{(t+l)^{m-q}}=\sum_{p=0}^{m}(-1)^p\binom{m}{p}\frac{(t+l)^{m-p}(l-k)^p}{(t+l)^{m-q}}\\ &=\sum_{p=0}^{m}(-1)^p\binom{m}{p}(l-k)^p(t+l)^{q-p},\\ &\qquad q=0,1,\dots,m-1,\quad k,l=0,\pm1,\dots,\pm n,\end{aligned}$$

we have for the integers $j \geqslant m$ that

$$\frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}t^j}\biggl((t+k)^m\frac{1}{(t+l)^{m-q}}\biggr)\bigg|_{t=-k}$$

$$= \biggl(\sum_{p=0}^{m}(-1)^p\binom{m}{p}\frac{(q-p)(q-p-1)\cdots(q-p-j+1)}{j!}$$

$$\times (l-k)^p(t+l)^{q-p-j}\biggr)\bigg|_{t=-k}$$

$$= \frac{1}{(l-k)^{j-q}}\sum_{p=0}^{m}(-1)^{p+j}\binom{m}{p}\frac{(p-q)_j}{j!} \tag{4.5}$$

if $l \neq k$ and

$$\frac{1}{j!}\frac{\mathrm{d}^j}{\mathrm{d}t^j}\biggl((t+k)^m\frac{1}{(t+l)^{m-q}}\biggr) = 0$$

if $l = k$. Since

$$\frac{D_n^{j-q}}{(l-k)^{j-q}} \in \mathbb{Z} \qquad \text{and} \qquad \binom{m}{p}\frac{(p-q)_j}{j!} \in \mathbb{Z},$$
$$q = 0, 1, \dots, m-1, \quad p = 0, 1, \dots, m, \quad k, l = 0, \pm 1, \dots, \pm n, \ k \neq l,$$

the inclusions (4.4) follow from (4.5), which completes the proof of the lemma.

The next lemma strengthens the inclusions (1.17). It will be used in our exposition of the general case.

Lemma 4.2. *We denote by $\mathcal{E} = \mathcal{E}_{n,m}$ the set of primes dividing each of the numbers*

$$B_{k,0} = \bigl(F(t)(t+k)^2\bigr)\big|_{t=-k} = \frac{(m+2n\pm k)!}{(m\pm k)!\,(n\pm k)!^2}, \qquad k = 0, \pm 1, \dots, \pm n,$$

where the function $F(t)$ is defined in (1.15). *Let*

$$\Pi = \Pi_{n,m} = \prod_{\substack{p\in\mathcal{E} \\ \sqrt{m+3n}<p\leqslant 2n}} p. \tag{4.6}$$

Then

$$\Pi^{-1}\frac{D_{2n}^j}{j!}\frac{\mathrm{d}^j}{\mathrm{d}t^j}\bigl(F(t)(t+k)^2\bigr)\bigg|_{t=-k} \in \mathbb{Z}, \qquad k = 0, \pm 1, \dots, \pm n, \tag{4.7}$$

for all non-negative integers j.

Proof. The fact that $\Pi^{-1}B_{k,0} \in \mathbb{Z}$, $k = 0, \pm 1, \dots, \pm n$ (inclusions (4.7) with $j = 0$) follows from the definition of $\mathcal{E}$ and Π. Let us verify inclusions (4.7) with $j = 1$.

We have

$$\begin{aligned} B_{k,1} &= \frac{\mathrm{d}}{\mathrm{d}t}\big(F(t)(t+k)^2\big)\bigg|_{t=-k} \\ &= B_{k,0}\Biggl(\sum_{l=m+1}^{m+2n}\biggl(\frac{1}{t+l}+\frac{1}{t-l}\biggr)-2\sum_{\substack{l=-n\\ l\ne k}}^{n}\frac{1}{t+l}\Biggr)\Bigg|_{t=-k} \\ &= B_{k,0}\Biggl(\sum_{l=m+1}^{m+2n}\biggl(\frac{1}{l-k}-\frac{1}{l+k}\biggr)-2\sum_{\substack{l=-n\\ l\ne k}}^{n}\frac{1}{l-k}\Biggr), \qquad k=0,\pm1,\dots,\pm n. \end{aligned} \tag{4.8}$$

By Lemma 1.3, we have $D_{2n}B_{k,1}\in\mathbb{Z}$, $k=0,\pm1,\dots,\pm n$. Therefore,

$$\operatorname{ord}_p(\Pi^{-1}D_{2n}B_{k,1})=\operatorname{ord}_p(D_{2n}B_{k,1})\geqslant 0 \tag{4.9}$$

if p is coprime with Π. Now assume that a prime p divides Π, whence

$$\operatorname{ord}_p\Pi=1. \tag{4.10}$$

For $p>\sqrt{m+3n}$ we have $\operatorname{ord}_p(l\pm k)\leqslant 1$, where $(l\pm k)$ is any denominator in (4.8). Therefore,

$$\operatorname{ord}_p\Biggl(\sum_{l=m+1}^{m+2n}\biggl(\frac{1}{l-k}-\frac{1}{l+k}\biggr)-2\sum_{\substack{l=-n\\ l\ne k}}^{n}\frac{1}{l-k}\Biggr)\geqslant -1, \qquad k=0,\pm1,\dots,\pm n. \tag{4.11}$$

We have

$$\operatorname{ord}_p D_{2n}\geqslant 1 \tag{4.12}$$

for $p\leqslant 2n$. Finally,

$$\operatorname{ord}_p B_{k,0}\geqslant 1, \qquad k=0,\pm1,\dots,\pm n, \tag{4.13}$$

if $p\in\mathcal{E}$.

Combining estimates (4.10)–(4.13) and taking (4.8) into account in the case when p divides Π, we obtain that

$$\operatorname{ord}_p(\Pi^{-1}D_{2n}B_{k,1})\geqslant 0. \tag{4.14}$$

By (4.9) and (4.14), we have $\Pi^{-1}D_{2n}B_{k,1}\in\mathbb{Z}$, $k=0,\pm1,\dots,\pm n$, which proves that (4.7) holds for $j=1$. In the case when $j\geqslant 2$ it remains to use Lemma 4.1 with $m=2$.

Note that under the assumptions of Lemma 4.2 we have

$$\operatorname{ord}_p B_{k,0}=\biggl\lfloor\frac{m+2n}{p}\pm\frac{k}{p}\biggr\rfloor-\biggl\lfloor\frac{m}{p}\pm\frac{k}{p}\biggr\rfloor-2\biggl\lfloor\frac{n}{p}\pm\frac{k}{p}\biggr\rfloor, \qquad k=0,\pm1,\dots,\pm n, \tag{4.15}$$

for any prime $p>\sqrt{m+3n}$, since

$$\operatorname{ord}_p q! = \biggl\lfloor\frac{q}{p}\biggr\rfloor+\biggl\lfloor\frac{q}{p^2}\biggr\rfloor+\biggl\lfloor\frac{q}{p^3}\biggr\rfloor+\cdots, \qquad q=1,2,\dots, \tag{4.16}$$

where $\lfloor\cdot\rfloor$ stands for the integer part of a number. Formula (4.15) enables us to calculate the asymptotics of the factor (4.6) as $n,m\to\infty$, since

$$\{p\in\mathcal{E}:p>\sqrt{m+3n}\,\}=\Bigl\{p>\sqrt{m+3n}:\min_{k=0,\pm1,\dots,\pm n}\{\operatorname{ord}_p B_{k,0}\}\geqslant 1\Bigr\}. \tag{4.17}$$

Consider, for example, the simplest version of (1.15) with $m=n$.

Lemma 4.3.

$$\varpi_1 = \lim_{n\to\infty} \frac{\log \Pi_{n,n}}{n} = \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{3}\right) + \psi(1) - \psi\left(\frac{5}{6}\right) - 1 \approx 0.4820, \tag{4.18}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ *is the logarithmic derivative of the gamma function.*

Proof. In the case when $m = n$ and $p > 2\sqrt{n}$, formula (4.15) implies that

$$\min_{k=0,\pm1,\dots,\pm n} \{\operatorname{ord}_p B_{k,0}\} = \min_{|k/p|\leqslant|n/p|} \varphi_1\left(\frac{n}{p}, \frac{k}{p}\right), \tag{4.19}$$

where

$$\varphi_1(x,y) = \lfloor 3x+y \rfloor + \lfloor 3x-y \rfloor - 3\lfloor x+y \rfloor - 3\lfloor x-y \rfloor. \tag{4.20}$$

It is obvious that the function (4.20) is periodic (with period 1) in each argument:

$$\varphi_1(x,y) = \varphi_1(\{x\},\{y\}), \qquad \{x\} = x - \lfloor x \rfloor.$$

Therefore, it is sufficient to calculate the values of this function inside the unit square. By the definition of the integer part of a number, the summands on the right-hand side of (4.20) change their values only when the point passes through the lines $3x \pm y = \text{const} \in \mathbb{Z}$ and $x \pm y = \text{const} \in \mathbb{Z}$ (see Fig. 9). This enables us to obtain the values of $\varphi_1(x,y)$ shown in Fig. 10.

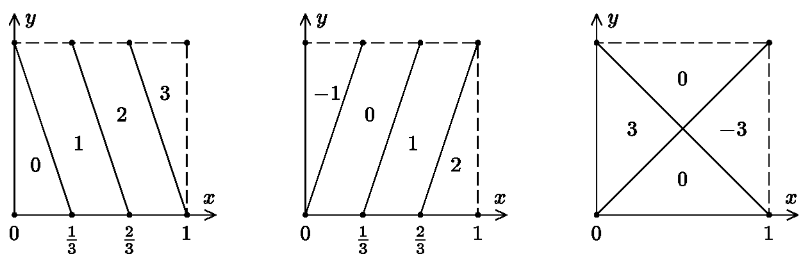

Figure 9. The values of the functions $\lfloor 3x+y \rfloor$, $\lfloor 3x-y \rfloor$ and $-3(\lfloor x+y \rfloor + \lfloor x-y \rfloor)$ inside the unit square

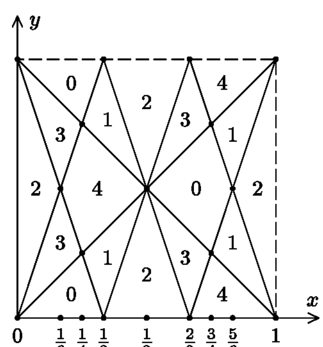

Figure 10. The values of the function $\varphi_1(x,y)$ inside the unit square

Consider the perpendiculars to the x-axis passing through the points of intersection of the lines shown in Fig. 10. We have

$$\min_{|y|\leqslant x}\varphi_1(x,y)=\min_{0\leqslant y\leqslant 1}\varphi_1(\{x\},y)=\begin{cases}1 & \text{if } \{x\}\in E_1=\left[\frac{1}{3},\frac{1}{2}\right)\cup\left[\frac{5}{6},1\right),\\ 0 & \text{otherwise.}\end{cases} \tag{4.21}$$

Formulae (4.17), (4.19) and (4.21) imply that

$$\{p\in\mathcal{E}_{n,n}:p>2\sqrt{n}\,\}=\left\{p>2\sqrt{n}:\left\{\frac{n}{p}\right\}\in E_1\right\},$$

whence

$$\Pi_{n,n}=\prod_{\substack{p:\{n/p\}\in E_1\\ 2\sqrt{n}<p\leqslant 2n}}p=\prod_{\substack{p:\{n/p\}\in E_1\\ p>2\sqrt{n}}}p\Big/\prod_{2n<p\leqslant 3n}p. \tag{4.22}$$

We have the following formula for the denominator of the fraction on the right-hand side of (4.22):

$$\prod_{2n<p\leqslant 3n}p=\frac{D_{3n}}{D_{2n}}. \tag{4.23}$$

We can find asymptotics of the numerator using following lemma (cf. [25], Theorem 4.3 and § 6, and [26], Lemma 3.2).

Lemma 4.4. *The limiting relation*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\substack{p>\sqrt{Cn}\\ \{n/p\}\in[u,v)}}\log p=\psi(v)-\psi(u)$$

holds for any $C>1$ *and any interval* $[u,v)\subset(0,1)$.

Using Lemma 4.4 and relations (4.22), (4.23) and (1.24), we obtain the desired asymptotics (4.18), which completes the proof of Lemma 4.3.

Remark 4.1. Simple as it is, the calculation of (4.21) can be made four times simpler. It is easy to verify that function (4.20) is not only periodic, but invariant under the transformations

$$\vartheta_1\colon (x,y)\mapsto\left(x+\frac{1}{2},y+\frac{1}{2}\right),\qquad \vartheta_2\colon (x,y)\mapsto(x,1-y) \tag{4.24}$$

(in the last map we use the shift by 1 and the fact that $\varphi_1(x,y)$ is odd with respect to y). Therefore, it is sufficient to find the values of the function (4.20) on the square $0\leqslant x,y<\frac{1}{2}$ and then use the transformations $\vartheta_1,\vartheta_1\circ\vartheta_2$ and ϑ_2 to extend this function to the unit square and so by periodicity to the whole of $\mathbb{R}^2$.

The next object of our study is the rational function

$$G(t)=G_n(t):=\frac{(t\pm(n+1))\cdots(t\pm(n+2rn))}{\big(t(t\pm1)\cdots(t\pm n)\big)^{2r}}. \tag{4.25}$$

Lemma 1.3 and the Leibniz rule for differentiating a product imply that

$$\frac{D_{2n}^j}{j!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\big(G(t)(t+k)^{2r}\big)\Big|_{t=-k}\in\mathbb{Z},\qquad k=0,\pm1,\dots,\pm n,\quad j=0,1,2,\dots. \tag{4.26}$$

Lemma 4.5. *For every integer* $r \geqslant 1$ *there is a sequence of integers* $\Pi_n = \Pi_n^{(r)} \geqslant 1$, $n = 1, 2, \dots,$ *such that*

$$\Pi_n^{-1} \frac{D_{2n}^j}{j!} \frac{\mathrm{d}^j}{\mathrm{d}t^j} \big(G(t)(t+k)^{2r}\big)\bigg|_{t=-k} \in \mathbb{Z}, \qquad k = 0, \pm 1, \dots, \pm n, \tag{4.27}$$

for all non-negative integers j, *and*

$$\varpi_r = \lim_{n\to\infty} \frac{\log \Pi_n^{(r)}}{n} = -2\sum_{l=1}^{r}\left(\psi\left(\frac{l}{r}\right) + \psi\left(\frac{l}{r+1/2}\right)\right) - (4r+1)\sum_{l=1}^{r}\frac{1}{l} - 4r\gamma + 4r, \tag{4.28}$$

where $\gamma \approx 0.57712$ *is Euler's constant.*

Proof. For $n \leqslant 2r$ we put $\Pi_n = 1$. Then the inclusions (4.27) follow from (4.26). This finite part of the sequence has no influence on the limit (4.28). In what follows we assume that $n > 2r$.

For every prime p we consider the quantity

$$\nu_p = \min_{k=0,\pm1,\dots,\pm n}\left\{\operatorname{ord}_p \frac{((2r+1)n+k)!\,((2r+1)n-k)!}{(n+k)!^{2r+1}(n-k)!^{2r+1}}\right\}. \tag{4.29}$$

Put

$$\Pi_n = \prod_{p:\sqrt{(2r+2)n}<p\leqslant 2n} p^{\nu_p}. \tag{4.30}$$

We fix an integer k in the interval $|k| \leqslant n$ and consider the function

$$G_k(t) := G(t)(t+k)^{2r}. \tag{4.31}$$

We claim that

$$\Pi_n^{-1} \frac{D_{2n}^j}{j!} \frac{\mathrm{d}^j G_k(t)}{\mathrm{d}t^j}\bigg|_{t=-k} \in \mathbb{Z}, \qquad j = 0, 1, 2, \dots. \tag{4.32}$$

By Lemma 4.1, it is sufficient to prove the inclusions (4.32) for $j \leqslant 2r$.

If p is a prime that does not divide Π_n, then (4.26) implies that

$$\operatorname{ord}_p\left(\Pi_n^{-1} \frac{D_{2n}^j}{j!} \frac{\mathrm{d}^j G_k(t)}{\mathrm{d}t^j}\bigg|_{t=-k}\right) = \operatorname{ord}_p\left(\frac{D_{2n}^j}{j!} \frac{\mathrm{d}^j G_k(t)}{\mathrm{d}t^j}\bigg|_{t=-k}\right) \geqslant 0. \tag{4.33}$$

Now assume that p is a prime dividing Π_n. We shall prove by induction on $j = 0, 1, \dots, 2r$ that

$$\operatorname{ord}_p\left(\Pi_n^{-1} \frac{D_{2n}^j}{j!} \frac{\mathrm{d}^j G_k(t)}{\mathrm{d}t^j}\bigg|_{t=-k}\right) \geqslant 0. \tag{4.34}$$

For $j = 0$ relation (4.34) follows from the definition of Π_n, since

$$G_k(t)\big|_{t=-k} = \frac{((2r+1)n+k)!\,((2r+1)n-k)!}{(n+k)!^{2r+1}(n-k)!^{2r+1}}.$$

We claim that (4.34) holds for $j+1$ if it holds for all preceding j. We shall use the following notation for the logarithmic derivative of the function (4.31):

$$g_k(t)=\frac{G_k'(t)}{G_k(t)}=\sum_{\substack{l=-(2r+1)n\\ l\neq k}}^{(2r+1)n}\frac{1}{t+l}-(2r+1)\sum_{\substack{l=-n\\ l\neq k}}^{n}\frac{1}{t+l}.$$

We have

$$\begin{aligned}\frac{1}{(j+1)!}\,\frac{\mathrm{d}^{j+1}G_k(t)}{\mathrm{d}t^{j+1}}&=\frac{1}{(j+1)!}\,\frac{\mathrm{d}^j}{\mathrm{d}t^j}\bigl(g_k(t)G_k(t)\bigr)\\ &=\frac{1}{j+1}\sum_{m=0}^{j}\frac{1}{(j-m)!}\,\frac{\mathrm{d}^{j-m}g_k(t)}{\mathrm{d}t^{j-m}}\cdot\frac{1}{m!}\,\frac{\mathrm{d}^m G_k(t)}{\mathrm{d}t^m}.\end{aligned} \tag{4.35}$$

We also have

$$\operatorname{ord}_p\frac{1}{j+1}=0, \tag{4.36}$$

since $p>\sqrt{(2r+2)n}>2r+1\geqslant j+1$. Further, we have

$$\begin{aligned}&\operatorname{ord}_p\biggl(\frac{1}{(j-m)!}\,\frac{\mathrm{d}^{j-m}g_k(t)}{\mathrm{d}t^{j-m}}\bigg|_{t=-k}\biggr)\\ &\quad=\operatorname{ord}_p\Biggl(\sum_{\substack{l=-(2r+1)n\\ l\neq k}}^{(2r+1)n}\frac{(-1)^{j-m}}{(l-k)^{j-m+1}}-(2r+1)\sum_{\substack{l=-n\\ l\neq k}}^{n}\frac{(-1)^{j-m}}{(l-k)^{j-m+1}}\Biggr)\\ &\quad\geqslant-(j-m+1)\end{aligned} \tag{4.37}$$

for $m<j$, since $p>\sqrt{(2r+2)n}$ and $|l-k|\leqslant(2r+2)n$ for all denominators in (4.37). The inequality

$$\operatorname{ord}_p D_{2n}^{j-m+1}\geqslant j-m+1 \tag{4.38}$$

holds, since $p\leqslant 2n$. Finally,

$$\operatorname{ord}_p\biggl(\Pi_n^{-1}\,\frac{D_{2n}^m}{m!}\,\frac{\mathrm{d}^m G_k(t)}{\mathrm{d}t^m}\bigg|_{t=-k}\biggr)\geqslant 0 \tag{4.39}$$

by the induction hypothesis. Substituting $t=-k$ into (4.35) and using estimates (4.36)–(4.39), we obtain that relation (4.34) holds for $j+1$. This completes the justification of the induction step.

Combining estimates (4.33) for $p\nmid\Pi_n$ and (4.34) for $p\mid\Pi_n$, we obtain that the inclusions (4.32) and (4.27) hold for $j=0,1,\dots,2r$. Therefore, they hold for all non-negative integers j. We claim that the limiting relation (4.28) holds for the Π_n, $n=1,2,\dots$.

By (4.29) and (4.16),

$$\nu_p = \min_{|k/p|\leqslant|n/p|} \varphi_r\left(\frac{n}{p}, \frac{k}{p}\right) \tag{4.40}$$

for every integer $n > 2r$ and prime $p > \sqrt{(2r+2)n}$, where the function

$$\varphi_r(x,y) = \lfloor(2r+1)x+y\rfloor+\lfloor(2r+1)x-y\rfloor-(2r+1)\lfloor x+y\rfloor-(2r+1)\lfloor x-y\rfloor \tag{4.41}$$

is periodic (with period 1) in each argument. We claim that

$$\min_{y\in\mathbb{R}} \varphi_r(x,y) = \nu \quad \text{if } \{x\} \in E_\nu, \qquad \nu = 0,1,\dots,2r-1, \tag{4.42}$$

where

$$\begin{aligned} E_{2l} &= \left[\frac{l}{2r}, \frac{l+1}{2r+1}\right) \cup \left[\frac{1}{2}+\frac{l}{2r}, \frac{1}{2}+\frac{l+1}{2r+1}\right), \qquad l = 0,1,\dots,r-1, \\ E_{2l-1} &= \left[\frac{l}{2r+1}, \frac{l}{2r}\right) \cup \left[\frac{1}{2}+\frac{l}{2r+1}, \frac{1}{2}+\frac{l}{2r}\right), \qquad l = 1,2,\dots,r. \end{aligned} \tag{4.43}$$

Taking into account that (4.41) is an odd function, we can easily show that this function is invariant under the transformations (4.24). Hence, relations (4.42) will be proved if we prove them in the domain $0 \leqslant x, y < \frac{1}{2}$. In this domain

$$\min_{0\leqslant y<1/2} \varphi_r(x,y) = \min_{0\leqslant y\leqslant x} \varphi_r(x,y) = \min_{0\leqslant y\leqslant x} \bigl(\lfloor(2r+1)x+y\rfloor + \lfloor(2r+1)x-y\rfloor\bigr),$$

since

$$-(2r+1)\lfloor x+y\rfloor - (2r+1)\lfloor x-y\rfloor = \begin{cases} 0 & \text{if } 0 \leqslant y \leqslant x < \frac{1}{2}, \\ 2r+1 & \text{if } 0 \leqslant x < y < \frac{1}{2}. \end{cases}$$

To complete the proof of relations (4.42), it remains to calculate the values of the function $\lfloor(2r+1)x+y\rfloor + \lfloor(2r+1)x-y\rfloor$ in the domain $0 \leqslant y \leqslant x < \frac{1}{2}$ (see Fig. 11) using arguments similar to those used in the proof of Lemma 4.3.

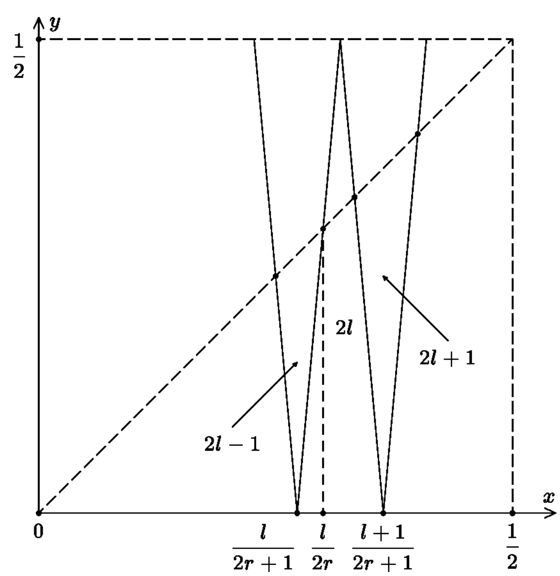

Figure 11. The values of the function $\varphi_r(x,y)$ in the domain $0 \leqslant y < x < \frac{1}{2}$

By (4.30), (4.40) and (4.42), (4.43), we have

$$\Pi_n = \prod_{\nu=1}^{2r-1} \prod_{\substack{p:\{n/p\}\in E_\nu \\ \sqrt{(2r+2)n}<p\leqslant 2n}} p^\nu$$
$$= \Biggl(\prod_{\nu=1}^{2r-1} \prod_{\substack{p:\{n/p\}\in E_\nu \\ p>\sqrt{(2r+2)n}}} p^\nu\Biggr) \Bigg/ \Biggl(\prod_{l=1}^{r-1} \prod_{\frac{2r+1}{l+1}n<p\leqslant\frac{2r}{l}n} p^{2l} \cdot \prod_{l=1}^{r} \prod_{\frac{2r}{l}n<p\leqslant\frac{2r+1}{l}n} p^{2l-1}\Biggr).$$

To obtain the limiting relation (4.28), we use Lemma 4.4, the identity

$$\psi(x)+\psi\biggl(x+\frac{1}{2}\biggr) = -2\log 2 + 2\psi(2x)$$

(see formula (4.47) below with $r=2$) and the relation

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\alpha n<p\leqslant\beta n}\log p = \beta-\alpha, \qquad \text{where } \alpha<\beta,$$

which follows from the prime number theorem (the asymptotic law of distribution of primes). We have

$$\begin{aligned}
\varpi_r &= \sum_{l=1}^{r-1} 2l\biggl(\psi\biggl(\frac{l+1}{2r+1}\biggr) - \psi\biggl(\frac{l}{2r}\biggr) + \psi\biggl(\frac{1}{2}+\frac{l+1}{2r+1}\biggr) - \psi\biggl(\frac{1}{2}+\frac{l}{2r}\biggr) \\
&\qquad - \biggl(\frac{2r}{l} - \frac{2r+1}{l+1}\biggr)\biggr) \\
&\quad + \sum_{l=1}^{r}(2l-1)\biggl(\psi\biggl(\frac{l}{2r}\biggr) - \psi\biggl(\frac{l}{2r+1}\biggr) + \psi\biggl(\frac{1}{2}+\frac{l}{2r}\biggr) - \psi\biggl(\frac{1}{2}+\frac{l}{2r+1}\biggr) \\
&\qquad - \biggl(\frac{2r+1}{l} - \frac{2r}{l}\biggr)\biggr) \\
&= 2\sum_{l=1}^{r-1} 2l\biggl(\psi\biggl(\frac{l+1}{r+1/2}\biggr) - \psi\biggl(\frac{l}{r}\biggr)\biggr) + 2\sum_{l=1}^{r}(2l-1)\biggl(\psi\biggl(\frac{l}{r}\biggr) - \psi\biggl(\frac{l}{r+1/2}\biggr)\biggr) \\
&\quad - \sum_{l=1}^{r-1} 2l\biggl(\frac{2r}{l} - \frac{2r+1}{l+1}\biggr) - \sum_{l=1}^{r}(2l-1)\biggl(\frac{2r+1}{l} - \frac{2r}{l}\biggr) \\
&= 2\sum_{l=1}^{r}\psi\biggl(\frac{l}{r}\biggr)(2l-1-2l) + 4r\psi(1) + 2\sum_{l=1}^{r}\psi\biggl(\frac{l}{r+1/2}\biggr)\bigl(2(l-1)-(2l-1)\bigr) \\
&\quad - \sum_{l=1}^{r}\frac{1}{l}\bigl(4rl - 2(l-1)(2r+1) + (2l-1)\bigr) + 4r \\
&= -2\sum_{l=1}^{r}\biggl(\psi\biggl(\frac{l}{r}\biggr) + \psi\biggl(\frac{l}{r+1/2}\biggr)\biggr) - (4r+1)\sum_{l=1}^{r}\frac{1}{l} + 4r\psi(1) + 4r.
\end{aligned}$$

Since $\psi(1) = -\gamma$ (see, for example, [27], § 1.1, formula (8)), we obtain the desired result, which completes the proof of the lemma.

Lemma 4.6. *The following estimates hold for the quantity defined by* (4.28):

$$-\log(r+1)-4<\varpi_r-4r+(4r+1)\gamma<\log(r+1)+\frac{1}{r}+2. \tag{4.44}$$

In particular,

$$\varpi_r=4r(1-\gamma)+O(\log r) \qquad \text{as } r\to\infty. \tag{4.45}$$

Proof. Since $\psi(x)$ is monotonically increasing on $(0,1]$, we have

$$-\psi(1)+\sum_{l=1}^{r+1}\psi\biggl(\frac{l}{r+1}\biggr)<\sum_{l=1}^{r}\psi\biggl(\frac{l}{r+1/2}\biggr)<\sum_{l=1}^{r}\psi\biggl(\frac{l}{r}\biggr). \tag{4.46}$$

We can calculate the sums in the upper and lower estimates using the formula

$$\sum_{l=1}^{r}\psi\biggl(x+\frac{l-1}{r}\biggr)=-r\log r+r\psi(rx) \tag{4.47}$$

(see, for example, [27], § 1.1, formula (6)).

Inequality (2.28) implies that the sequence

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{r}-\log r$$

decreases and the sequence

$$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{r}-\log(r+1)$$

increases. Therefore, their common limit γ lies between the elements of these sequences, that is,

$$\log r<\sum_{l=1}^{r}\frac{1}{l}-\gamma<\log(r+1). \tag{4.48}$$

Substituting estimates (4.46)–(4.48) into (4.28), we obtain that

$$-4r\log\frac{r+1}{r}-\log(r+1)<\varpi_r-4r+(4r+1)\gamma<(2r+1)\log\frac{r+1}{r}+\log(r+1).$$

Using inequality (2.28) once again, we complete the proof of estimates (4.44). The limiting relation (4.45) follows immediately from (4.44). The lemma is proved.

Proposition 4.1. *The denominator* $\operatorname{den}(I_n)$ *of the linear form* (1.5) *divides* $\Pi_n^{-b}D_{2n}^{a+b-1}$, *where the sequence of integers* $\Pi_n=\Pi_n^{(r)}$, $n=1,2,\dots$, *is defined in Lemma* 4.5. *Hence*,

$$\varlimsup_{n\to\infty}\frac{\log\operatorname{den}(I_n)}{n}\leqslant\lim_{n\to\infty}\frac{\log D_{2n}^{a+b-1}-b\log\Pi_n}{n}=2(a+b-1)-b\varpi_r.$$

Proof. We prove this proposition by replacing (1.22) in the proof of Lemma 1.4 by the representation

$$R(t) = H_n(t)^{a-2rb} G_n(t)^b,$$

applying Lemmas 1.3 and 4.5 to the functions (1.16) and (4.25), and using the Leibniz rule for the differentiation of a product.

To conclude this section we give some particular values of (4.28) (for $r = 1$ see (4.18)):

$$\varpi_2 \approx 2.01561, \quad \varpi_3 \approx 3.64442, \quad \varpi_{10} \approx 15.38202, \quad \varpi_{50} \approx 82.98948,$$
$$\varpi_{100} \approx 1.67541 \cdot 100, \quad \varpi_{1000} \approx 1.68956 \cdot 1000, \quad \varpi_r \approx 4(1-\gamma)r \approx 1.69114\, r.$$

§ 5. Proof of the results on linear independence

The following theorem is based on the results obtained in § 2 and § 4.

Theorem 5.1. *Let the assumptions of Proposition* 2.3 *hold*, *let* b *be an odd integer*, *and let*

$$\varkappa + 2(a+b-1) - b\varpi_r < 0,$$

where

$$\varpi_r = -2\sum_{l=1}^{r}\biggl(\psi\Bigl(\frac{l}{r}\Bigr) + \psi\Bigl(\frac{l}{r+1/2}\Bigr)\biggr) - (4r+1)\sum_{l=1}^{r}\frac{1}{l} - 4r\gamma + 4r.$$

Then at least one of the numbers

$$\zeta(b+2),\ \zeta(b+4),\ \dots,\ \zeta(a+b-1) \tag{5.1}$$

is irrational.

Proof. By Lemma 1.1, for any integer $n \geqslant 1$ the quantity defined by (1.2) is a linear form in 1 and the numbers (5.1). Assuming that every number in (5.1) is rational and denoting their least common denominator by D, we obtain that $J_n = D|I_n|\operatorname{den}(I_n)$ is a positive integer for infinitely many $n \geqslant 1$. This contradicts the limiting relation

$$\varlimsup_{n\to\infty} \frac{\log J_n}{n} \leqslant \varkappa + 2(a+b-1) - b\varpi_r < 0,$$

which follows from Propositions 2.3 and 4.1.

Proof of Theorem 0.1. We use Theorem 5.1 with suitable a, b and $r = 1$ for each set of numbers. Note that $\mu_1 < 3.05$ in each of the cases considered below. Hence, assumption (2.61) holds.

We prove that at least one of the elements of the first set in (0.1) is irrational by putting $a = 19$ and $b = 3$ in Theorem 5.1. In this case $\mu_1 \approx 3.04028$, $\tau_0 \approx 2.98027 + 0.02985i$, and $\operatorname{Im} f_0(\tau_0) + 3\pi \approx 0.09308 \not\equiv 0 \pmod{\pi\mathbb{Z}}$. Combining the estimate

$$\varkappa + 2(a+b-1) - b\varpi_r \approx -0.72567 < 0$$

with Theorem 5.1, we complete the proof of the assertion for the first set in (0.1).

For the second we put $a = 33$ and $b = 5$. Then

$$\mu_1 \approx 3.03309, \qquad \tau_0 \approx 3.00783 + 0.03046i, \qquad \operatorname{Im} f_0(\tau_0) + 9\pi \approx 0.14978,$$
$$\varkappa + 2(a + b - 1) - b\varpi_r \approx -0.76662 < 0.$$

Finally, putting $a = 47$ and $b = 7$ for the third, we obtain

$$\mu_1 \approx 3.03043, \qquad \tau_0 \approx 3.01730 + 0.02406i, \qquad \operatorname{Im} f_0(\tau_0) + 15\pi \approx 0.16232,$$
$$\varkappa + 2(a + b - 1) - b\varpi_r \approx -0.82928 < 0.$$

The theorem is proved.

Proof of Theorem 0.2. We put $a = 7b$ and $r = 1$ for every odd integer $b \geqslant 3$ and denote by τ_b the root of the corresponding polynomial (2.21) in the domain $\operatorname{Re}\tau > 0$, $\operatorname{Im}\tau > 0$ with the maximal possible $\operatorname{Re}\tau_b$. Note that the root of polynomial (2.21) in the interval $(3, +\infty)$ coincides with the root of the polynomial

$$(\tau + 3)(\tau - 1)^8 - (\tau - 3)(\tau + 1)^8,$$

which is equal to $\mu_1 \approx 3.02472$. Since (2.61) holds, Lemma 2.10 implies that $\operatorname{Re} f_0(\tau_b) \leqslant \operatorname{Re} f_0(\mu_1)$, whence

$$\varkappa \leqslant b\varkappa_0 := b \log \frac{2^{2(a-2)}(\mu_1 + 3)^3(\mu_1 - 3)^3}{(\mu_1 + 1)^8(\mu_1 - 1)^8} \approx -15.56497\, b.$$

Therefore,

$$\varkappa + 2(a + b - 1) - b\varpi_r < b\varkappa_0 + 16b - b\varpi_1 \approx -0.04701\, b < 0.$$

We shall be able to use Theorem 5.1 if we can show that (2.55) holds. It is easy to verify that

$$\begin{aligned} g_0(b) &= \operatorname{Im} f_0(\tau_b) + 3(b-2)\pi \\ &= 3b\bigl(\arg(\tau_b - 3) + \arg(-\tau_b + 3)\bigr) - 8b\bigl(\arg(\tau_b - 1) + \arg(\tau_b + 1)\bigr) + 3(b-2)\pi \\ &= 2b\bigl(3\arg(\tau_b - 3) + 8\arg(\tau_b - 1) - 16\arg(\tau_b + 1)\bigr), \end{aligned}$$

regarded as a function of $b \geqslant 3$, increases, whence $g_0(b) \geqslant g_0(3) \approx 0.05935 > 0$. The relations

$$\arg(-\tau_b + 3) \sim \frac{b-2}{b}\pi, \quad |-\tau_b + 3| \sim |\mu_1 - 3| \qquad \text{as } b \to \infty, \ b \text{ is odd},$$

enable us to calculate the limit

$$\lim_{\substack{b \to \infty \\ b \text{ is odd}}} g_0(b) < \pi.$$

Therefore, $\operatorname{Im} f_0(\tau_b) \not\equiv 0 \pmod{\pi\mathbb{Z}}$. An application of Theorem 5.1 completes the proof.

Proof of Theorem 0.3. By Lemma 1.1 and Proposition 4.1 with $b = 1$, the $\tilde{I}_n(a,r) = I_n D_{2n}^a / \Pi_n^{(r)}$ defined by (1.2) are linear forms in $1, \zeta(3), \zeta(5), \dots, \zeta(a)$ with integer coefficients. By Propositions 2.1, 3.1, and 4.1 the α, β in the criterion for linear independence (see the introduction) are defined by the formulae

$$\begin{aligned} \alpha(a,r) &= -\varkappa(a,r) - 2a + \varpi_r, \\ \beta(a,r) &= 2(2r+1)\log(2r+1) + 2(a-2r)\log 2 + 2a - \varpi_r, \end{aligned} \tag{5.2}$$

where $\varkappa = \varkappa(a,r)$ is defined by (2.59). For $a = 145$, $r = 10$ and $a = 1971$, $r = 65$ we have

$$\begin{aligned} \mu_1(145,10) - 21 &\approx 0.38013 \cdot 10^{-4}, & 1 + \frac{\alpha(145,10)}{\beta(145,10)} &\approx 2.000397, \\ \mu_1(1971,65) - 131 &\approx 0.22019 \cdot 10^{-10}, & 1 + \frac{\alpha(1971,65)}{\beta(1971,65)} &\approx 3.000103, \end{aligned}$$

where $\mu_1(a,r)$ is the real root of polynomial (2.21) with $b = 1$ in the interval $(2r+1, +\infty)$. The criterion for linear independence implies that

$$\delta(145) \geqslant 3, \qquad \delta(1971) \geqslant 4, \tag{5.3}$$

since the dimension of space is an integer. By Apéry's theorem, $\delta(3) = 2$. The assertion of the theorem follows from the estimates obtained.

Lemma 5.1. *Assume that the positive integers a, r are such that*

$$a > 2(r+2)\log(r+1) + 1, \tag{5.4}$$

and let $\mu_1 = \mu_1(a,r)$ be a real root of the polynomial

$$h(\tau) = (\tau + 2r + 1)(\tau - 1)^{a+1} - (\tau - 2r - 1)(\tau + 1)^{a+1} \tag{5.5}$$

in the interval $(2r+1, +\infty)$. Then

$$0 < \mu_1 - (2r+1) < 2\varepsilon, \qquad \textit{where} \quad \varepsilon := \frac{2r+1}{(r+1)^2} < 1. \tag{5.6}$$

Proof. Direct calculations show that

$$h(2r+1) = 2^{a+2}(r+1)r^{a+1} > 0, \tag{5.7}$$

$$h(2r+1+2\varepsilon) = 2^{a+2}\big((2r+1+\varepsilon)(r+\varepsilon)^{a+1} - \varepsilon(r+1+\varepsilon)^{a+1}\big). \tag{5.8}$$

Inequality (5.4) implies that

$$(r+1+\varepsilon)^{a-1} > (r+\varepsilon)^{a+1}, \tag{5.9}$$

since

$$\begin{aligned}
&(a-1)\log(r+1+\varepsilon)-(a+1)\log(r+\varepsilon)\\
&\qquad=(a-1)\log\left(1+\frac{1}{r+\varepsilon}\right)-2\log(r+\varepsilon)\\
&\qquad>(a-1)\log\left(1+\frac{1}{r+1}\right)-2\log(r+1)\\
&\qquad>\frac{a-1}{r+2}-2\log(r+1)=\frac{a-1-2(r+2)\log(r+1)}{r+2}>0.
\end{aligned}$$

Besides,

$$\begin{aligned}
\varepsilon(r+1+\varepsilon)^2&=\frac{(2r+1)(r+1+\varepsilon)^2}{(r+1)^2}\\
&>\frac{(2r+1)(r+1+\varepsilon)^2}{(r+1)^2}-\frac{\varepsilon(3r^2+4r+1+2\varepsilon r+\varepsilon)}{(r+1)^2}=2r+1+\varepsilon.
\end{aligned}\tag{5.10}$$

Substituting inequalities (5.9) and (5.10) into (5.8), we obtain that $h(2r+1+2\varepsilon)<0$. Combining this inequality with (5.7), we obtain that the interval $(2r+1, 2r+1+2\varepsilon)$ contains at least one root of the polynomial (5.5). Now (5.6) follows from the fact that μ_1 is the unique real root in the interval $(2r+1,+\infty)$ (Lemma 2.7). The lemma is proved.

Corollary 5.1. *Let the assumptions of Lemma* 5.1 *hold with* $r>6$. *Then the following estimate holds for* $\varkappa=\varkappa(a,r)$ *in* (2.59):

$$\varkappa(a,r)<2(2r+1)\log(2r+1)-2(a+1)(1+\log r).$$

Proof. Since $h(\mu_1)=0$, we have

$$\varkappa=\log\frac{2^{2(a-2r)}(\mu_1+2r+1)^{2(2r+1)}(\mu_1-1)^{2(a+1)r}}{(\mu_1+1)^{2(a+1)(r+1)}}.$$

Combining this with (5.6), we obtain that

$$\begin{aligned}
&\frac{\varkappa}{2}-(a-2r)\log 2\\
&\quad=(2r+1)\log\left(1+\frac{2r}{\mu_1+1}\right)-(a+1)r\log\left(1+\frac{2}{\mu_1-1}\right)-(a-2r)\log(\mu_1+1)\\
&\quad<(2r+1)\log\left(1+\frac{r}{r+1}\right)-\frac{(a+1)r}{r+1}-(a-2r)\log(2r+2)\\
&\quad=(2r+1)\log(2r+1)-(a+1)\left(\frac{r}{r+1}+\log(r+1)\right)-(a-2r)\log 2\\
&\quad<(2r+1)\log(2r+1)-(a+1)(1+\log r)-(a-2r)\log 2.
\end{aligned}$$

Here we have used the inequality

$$\log\left(1+\frac{2}{\mu_1-1}\right) > \log\left(1+\frac{1}{r+\varepsilon}\right) > 2\log\left(1+\frac{1}{2r+1}\right) > \frac{1}{r+1}$$

(see also (2.28) with $n=2r+1$), which holds for $r>6$ since

$$\begin{aligned} 1+\frac{1}{r+\varepsilon}-\left(1+\frac{1}{2r+1}\right)^2 &= \frac{r-4\varepsilon r-3r+1}{(r+\varepsilon)(2r+1)^2} \\ &= \frac{r^3-5r^2-7r-2}{(r+\varepsilon)(r+1)^2(2r+1)^2} > 0 \qquad \text{if } r>6. \end{aligned}$$

The corollary is proved.

Proof of Theorem 0.4. Since $\delta(a)$ is an integer and

$$0.395\cdot\log 3 \approx 0.43395, \quad 0.395\cdot\log 145 \approx 1.96581, \quad 0.395\cdot\log 1971 \approx 2.99659,$$

Apéry's theorem and estimates (5.3) imply that inequality (0.2) holds for all odd integers $a < e^{4/0.395}$, that is, $a < 24999$. In fact we shall show that the following estimate holds for odd integers $a \geqslant 20737 = 12^4+1$, which is stronger than (0.2):

$$\delta(a) > \frac{\log a}{\log 12}, \tag{5.11}$$

or, which is the same,

$$\delta(12^m+1) > m, \qquad m=4,5,6,\dots. \tag{5.12}$$

For every $a=12^m+1$ we put

$$r = \left\lfloor \frac{\log^2 12}{3}\cdot\frac{a}{\log^2 a}\right\rfloor = \left\lfloor \frac{12^m+1}{3m^2}\right\rfloor, \qquad m=4,5,6,\dots,$$

and obtain a lower estimate for $\delta(a)$ using the criterion for linear independence, formulae (5.2), and the inequalities in Lemma 4.6 and Corollary 5.1:

$$\begin{aligned} \delta(a) &\geqslant 1+\frac{\alpha(a,r)}{\beta(a,r)} \\ &> \frac{2(a+1)(1+\log r)+2(a-2r)\log 2}{2(2r+1)\log(2r+1)+2(a-2r)\log 2+2a-4r(1-\gamma)+\log(r+1)+4+\gamma}. \end{aligned} \tag{5.13}$$

Now for $m=4,5,6,7$ estimate (5.12) follows immediately from (5.13):

m	$a=12^m+1$	$r=\left\lfloor \frac{a}{3m^2}\right\rfloor$	$\delta(a)$
4	20737	432	> 4.00882
5	248833	3317	> 5.15339
6	2985985	27648	> 6.35168
7	35831809	243753	> 7.58967

For the remaining values of m we combine (5.13) with the trivial estimates

$$0<\frac{r}{a}\leqslant\frac{1}{3m^2},\qquad \frac{2r+1}{a}\leqslant\frac{2}{3m^2}+\frac{1}{a}\leqslant\frac{2}{m^2\log 12},$$
$$\log r\geqslant\log\frac{12^m}{4m^2}=m\log 12-2\log 2-2\log m,$$
$$\log(2r+1)\leqslant\log(a-1)=m\log 12.$$

We have

$$\delta(a)=\delta(12^m+1)>\frac{m\log 12-2\log m+1-\log 2-1/(3m^2)}{1+\log 2+2/m}>\frac{m\log 12-2\log m}{1+\log 2+2/m}.$$

Now (5.12) with $m\geqslant 8$ follows from the fact that the function

$$\Delta(m)=(m\log 12-2\log m)-m\biggl(1+\log 2+\frac{2}{m}\biggr)=m(\log 6-1)-2\log m-2$$

increases monotonically and takes a positive value, $\Delta(8)\approx 0.17519$, at $m=8$. This completes the proof of the theorem.

In fact we have proved estimate (5.11) for odd integers $a\geqslant 3$ with the exception of finitely many values in the range $12^3+1=1729\leqslant a\leqslant 1969$.

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Received 24/APR/01
Translated by V. M. MILLIONSHCHIKOV